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Local Homotopy Theory

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Contents

Part I
Preliminaries

This sequence of three chapters gives some background and notation from the homotopy theory of simplicial sets and topos theory.

There is also a preliminary discussion of Suslin's rigidity theorem in Algebraic K-theory and the generalized isomorphism conjecture for the cohomology of algebraic groups of Friedlander and Milnor. Together with the Lichtenbaum-Quillen conjecture, these were the original motivating principles and problems which led to the introduction and development of local homotopy theory in its present form.

Chapter 1
The homotopy theory of simplicial sets

1.1 Simplicial sets

The finite ordinal number $\mathbf{n}$ is the set of counting numbers

$$\mathbf{n} = \{0, 1, \ldots, n\}.$$

There is an obvious ordering on this set which gives it the structure of a poset, and hence a category. In general, if C is a category then the functors $\alpha : \mathbf{n} \to C$ can be identified with strings of arrows

$$\alpha(0) \to \alpha(1) \to \cdots \to \alpha(n)$$

of length n. The collection of all finite ordinal numbers and all order-preserving functions between them (aka. poset morphisms, or functors) form the *ordinal number category* $\boldsymbol{\Delta}$.

Example 1.1. The ordinal number monomorphisms $d^i : \mathbf{n-1} \to \mathbf{n}$ are defined by the strings of relations

$$0 \leq 1 \leq \cdots \leq i-1 \leq i+1 \leq \cdots \leq n$$

for $0 \leq i \leq n$. These morphisms are called *cofaces* .

Example 1.2. The ordinal number epimorphisms $s^j : \mathbf{n+1} \to \mathbf{n}$ are defined by the strings

$$0 \leq 1 \leq \cdots \leq j \leq j \leq \cdots \leq n$$

for $0 \leq j \leq n$. These are the *codegeneracies* .

The cofaces and codegeneracies together satisfy the following relations

$$
\begin{aligned}
d^j d^i &= d^i d^{j-1} \text{ if } i < j, \\
s^j s^i &= s^i s^{j+1} \text{ if } i \leq j \\
s^j d^i &= \begin{cases} d^i s^{j-1} & \text{if } i < j, \\ 1 & \text{if } i = j, j+1, \\ d^{i-1} s^j & \text{if } i > j+1. \end{cases}
\end{aligned}
\tag{1.1}
$$

The ordinal number category $\boldsymbol{\Delta}$ is the category which is generated by the cofaces and codegeneracies, subject to the *cosimplicial identities* (1.1) [51] . Every ordinal number morphism has a unique epi-monic factorization, and has a canonical form defined in terms of strings of codegeneracies and strings of cofaces.

A *simplicial set* is a functor $X : \boldsymbol{\Delta}^{op} \to \mathbf{Set}$, ie. a contravariant set-valued functor on the ordinal number category $\boldsymbol{\Delta}$. Such things are usually written $\mathbf{n} \mapsto X_n$, and X_n is called the set of *n-simplices* of X. A *simplicial set map* (or *simplicial map*) $f : X \to Y$ is a natural transformation of such functors. The simplicial sets and simplicial maps form the category of simplicial sets, which will be denoted by $s\mathbf{Set}$.

A simplicial set is a simplicial object in the set category. Generally, $s\mathbf{A}$ denotes the category of simplicial objects in a category $\mathbf{A}$. Examples include the categories $s\mathbf{Gr}$ of simplicial groups, $s(R-\mathbf{Mod})$ of simplicial R-modules, $s(s\mathbf{Set}) = s^2\mathbf{Set}$ of bisimplicial sets, and so on.

Example 1.3. The topological standard n-simplex is the space

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum_{i=0}^{n} t_i = 1\}$$

The assignment $\mathbf{n} \mapsto |\Delta^n|$ is a cosimplicial space, or cosimplicial object in spaces. A covariant functor $\boldsymbol{\Delta} \to \mathbf{A}$ is a *cosimplicial object* in the category $\mathbf{A}$.

If X is a topological space, then the *singular set* or *singular complex* $S(X)$ is the simplicial set which is defined by

$$S(X)_n = \hom(|\Delta^n|, X).$$

The assignment $X \mapsto S(X)$ defines a covariant functor

$$S : \mathbf{CGHaus} \to s\mathbf{Set},$$

and this functor is called the *singular functor*. Here, **CGHaus** is the category of compactly generated Hausdorff spaces, which is the “good” category of spaces for homotopy theory [24, I.2.4].

Example 1.4. The ordinal number $\mathbf{n}$ represents a contravariant functor

$$\Delta^n = \hom_{\boldsymbol{\Delta}}(\ ,\mathbf{n}),$$

which is called the *standard n-simplex* . Write

$$\iota_n = 1_{\mathbf{n}} \in \hom_{\boldsymbol{\Delta}}(\mathbf{n}, \mathbf{n}).$$

The n-simplex ι_n is often called the *classifying n-simplex*, because the Yoneda Lemma implies that there is a natural bijection

$$\hom_{s\mathbf{Set}}(\Delta^n, Y) \cong Y_n$$

defined by sending the map $\sigma : \Delta^n \to Y$ to the element $\sigma(\iota_n) \in Y_n$. I usually say that a simplicial set map $\Delta^n \to Y$ is an n-simplex of Y.

In general, if $\sigma : \Delta^n \to X$ is a simplex of X, then the i^{th} *face* $d_i(\sigma)$ is the composite

$$\Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{\sigma} X,$$

while the j^{th} *degeneracy* $s_j(\sigma)$ is the composite

$$\Delta^{n+1} \xrightarrow{s^j} \Delta^n \xrightarrow{\sigma} X.$$

Example 1.5. The simplicial set $\partial\Delta^n$ is the subobject of Δ^n which is generated by the $(n-1)$-simplices d^i, $0 \leq i \leq n$, and Λ^n_k is the subobject of $\partial\Delta^n$ which is generated by the simplices d^i, $i \neq k$. The object $\partial\Delta^n$ is called the *boundary* of Δ^n, and Λ^n_k is called the k^{th} *horn*.

The faces $d^i : \Delta^{n-1} \to \Delta^n$ determine a covering

$$\bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n,$$

and for each $i < j$ there are pullback diagrams

$$\begin{tikzcd} \Delta^{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] & \Delta^{n-1} \arrow[d, "d^i"] \\ \Delta^{n-1} \arrow[r, "d^j"'] & \Delta^n \end{tikzcd}$$

It follows that there is a coequalizer

$$\bigsqcup_{i<j,0\leq i,j\leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{0\leq i\leq n} \Delta^{n-1} \longrightarrow \partial\Delta^n$$

in $s\mathbf{Set}$. Similarly, there is a coequalizer

$$\bigsqcup_{i<j,i,j\neq k} \Delta^{n-2} \rightrightarrows \bigsqcup_{0\leq i\leq n,i\neq k} \Delta^{n-1} \longrightarrow \Lambda^n_k.$$

Example 1.6. Suppose that a category C is *small* in the sense that the morphisms $\mathrm{Mor}(C)$ is a set. Examples of such things include all finite ordinal numbers $\mathbf{n}$, all monoids (small categories having one object), and all groups.

If C is a small category there is a simplicial set BC with

$$BC_n = \hom(\mathbf{n}, C),$$

meaning the functors $\mathbf{n} \to C$. The simplicial structure on BC is defined by precomposition with ordinal number maps. The object BC is called, variously, the *classifying space* or *nerve* of C.

Note that $B\mathbf{n} = \Delta^n$ in this notation.

Example 1.7. Suppose that I is a small category, and that $X : I \to \mathbf{Set}$ is a set-valued functor. The *category of elements* (or *translation category*, or *slice category*)

$$*/X = E_I(X)$$

associated to X has as objects all pairs (i,x) with $x \in X(i)$, or equivalently all functions

$$* \xrightarrow{x} X(i).$$

A morphism $\alpha : (i,x) \to (j,y)$ is a morphism $\alpha : i \to j$ of I such that $\alpha_*(x) = y$, or equivalently a commutative diagram

$$\begin{tikzcd} & X(i) \arrow[dd, "\alpha_*"] \\ * \arrow[ur, "x"] \arrow[dr, "y"'] & \\ & X(j) \end{tikzcd}$$

The simplicial set $B(E_IX)$ is often called the *homotopy colimit* for the functor X, and one writes

$$\underrightarrow{\mathrm{holim}}_I\, X = B(E_IX).$$

There is a canonical functor $E_IX \to I$ which is defined by the assignment $(i,x) \mapsto i$, which induces a canonical simplicial set map

$$\pi : B(E_IX) = \underrightarrow{\mathrm{holim}}_I\, X \to BI.$$

The functors $\mathbf{n} \to E_IX$ can be identified with strings

$$(i_0,x_0) \xrightarrow{\alpha_1} (i_1,x_1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} (i_n,x_n).$$

Note that such a string is uniquely specified by the underlying string $i_0 \to \cdots \to i_n$ in the index category Y and $x_0 \in X(i_0)$. It follows that there is an identification

$$(\underrightarrow{\mathrm{holim}}_I\, X)_n = B(E_IX)_n = \bigsqcup_{i_0 \to \cdots \to i_n} X(i_0).$$

This construction is natural with respect to natural transformations in X. Thus a diagram $Y : I \to s\mathbf{Set}$ in simplicial sets determines a bisimplicial set with (n,m) simplices

$$B(E_IY)_m = \bigsqcup_{i_0 \to \cdots \to i_n} Y(i_0)_m.$$

The *diagonal* $d(Z)$ of a bisimplicial set Z is the simplicial set with n-simplices $Z_{n,n}$. Equivalently, $d(Z)$ is the composite functor

$$\mathbf{\Delta}^{op} \xrightarrow{\Delta} \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \xrightarrow{Z} \mathbf{Set}$$

where Δ is the diagonal functor.

The diagonal $dB(E_I Y)$ of the bisimplicial set $B(E_I Y)$ is the homotopy colimit $\underrightarrow{\mathrm{holim}}_I\, Y$ of the diagram $Y : I \to s\mathbf{Set}$ in simplicial sets. There is a natural simplicial set map

$$\pi : \underrightarrow{\mathrm{holim}}_I\, Y \to BI.$$

Example 1.8. Suppose that X and Y are simplicial sets. There is a simplicial set $\mathbf{hom}(X,Y)$ with n-simplices

$$\mathbf{hom}(X,Y)_n = \hom(X \times \Delta^n, Y),$$

called the *function complex* .

There is a natural simplicial set map

$$ev : X \times \mathbf{hom}(X,Y) \to Y$$

which sends the pair $(x, f : X \times \Delta^n \to Y)$ to the simplex $f(x, \iota_n)$. Suppose that K is another simplicial set. The function

$$ev_* : \hom(K, \mathbf{hom}(X,Y)) \to \hom(X \times K, Y),$$

which is defined by sending the map $g : K \to \mathbf{hom}(X,Y)$ to the composite

$$X \times K \xrightarrow{1\times g} X \times \mathbf{hom}(X,Y) \xrightarrow{ev} Y,$$

is a natural bijection, giving the *exponential law*

$$\hom(K, \mathbf{hom}(X,Y)) \cong \hom(X \times K, Y).$$

This natural isomorphism gives $s\mathbf{Set}$ the structure of a cartesian closed category. The function complexes also give $s\mathbf{Set}$ the structure of a category enriched in simplicial sets.

1.2 The simplex category and realization

The *simplex category* $\mathbf{\Delta}/X$ for a simplicial set X has for objects all simplices $\Delta^n \to X$. Its morphisms are the incidence relations between the simplices, meaning all commutative diagrams

$$\begin{tikzcd} \Delta^m \arrow[d, "\theta"'] \arrow[dr, "\tau"] & \\ \Delta^n \arrow[r, "\sigma"'] & X \end{tikzcd} \tag{1.2}$$

The *realization* $|X|$ of a simplicial set X is defined by

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|,$$

meaning that the space $|X|$ is constructed by glueing together copies of the spaces described in Example 1.3 along the incidence relations of the simplices of X. The assignment $X \mapsto |X|$ defines a functor

$$|\ | : s\mathbf{Set} \to \mathbf{CGHaus}.$$

The proof of the following lemma is an exercise:

Lemma 1.9. *The realization functor $|\ |$ is left adjoint to the singular functor S.*

Example 1.10. The realization $|\Delta^n|$ of the standard n-simplex is the space $|\Delta^n|$ described in Example 1.3, since the simplex category $\mathbf{\Delta}/\Delta^n$ has a terminal object, namely $1 : \Delta^n \to \Delta^n$.

Example 1.11. The realization $|\partial\Delta^n|$ of the simplicial set $\partial\Delta^n$ is the topological boundary $\partial|\Delta^n|$ of the space $|\Delta^n|$. The space $|\Lambda^n_k|$ is the part of the boundary $\partial|\Delta^n|$ with the face opposite the vertex k removed. In effect, the realization functor is a left adjoint and therefore preserves coequalizers and coproducts.

The *n-skeleton* $\mathrm{sk}_n X$ of a simplicial set X is the subobject generated by the simplices X_i, $0 \leq i \leq n$. The ascending sequence of subcomplexes

$$\mathrm{sk}_0 X \subset \mathrm{sk}_1 X \subset \mathrm{sk}_2 X \subset \dots$$

defines a filtration of X, and there are pushout diagrams

$$\begin{tikzcd} \bigsqcup_{x \in NX_n} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} X \arrow[d] \\ \bigsqcup_{x \in NX_n} \Delta^n \arrow[r] & \mathrm{sk}_n X \end{tikzcd}$$

Here, NX_n denotes the set of non-degenerate n-simplices of X.

It follows that the realization of a simplicial set is a CW-complex. Every monomorphism $A \to B$ of simplicial sets induces a cofibration $|A| \to |B|$ of spaces, since $|B|$ is constructed from $|A|$ by attaching cells.

The realization functor preserves colimits (is right exact) because it has a right adjoint. The realization functor, when interpreted as taking values in compactly generated Hausdorff spaces, also has a fundamental left exactness property:

Lemma 1.12. *The realization functor*

$$| \ | : s\mathbf{Set} \to \mathbf{CGHaus}.$$

preserves finite limits. Equivalently, it preserves finite products and equalizers.

This result is proved in [20].

1.3 Model structure for simplicial sets

This section summarizes material which is presented in some detail in [24].

Say that a map $f : X \to Y$ of simplicial sets is a *weak equivalence* if the induced map $f_* : |X| \to |Y|$ is a weak equivalence of **CGHaus**. A map $i : A \to B$ of simplicial sets is a *cofibration* if and only if it is a monomorphism, meaning that all functions $i : A_n \to B_n$ are injective. A simplicial set map $p : X \to Y$ is a *fibration* if and only if it has the right lifting property with respect to all trivial cofibrations.

As usual, a *trivial cofibration* (respectively *trivial fibration*) is a cofibration (respectively fibration) which is also a weak equivalence.

Throughout this book, a *closed model category* will be a category **M** equipped with three classes of maps, called cofibrations, fibrations and weak equivalences such that the following axioms are satisfied:

CM1 The category **M** has all finite limits and colimits.
CM2 Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

in **M**. If any two of the maps f, g and h are weak equivalences, then so is the third.
CM3 If a map f us a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.
CM4 Suppose given a commutative solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration. Then the dotted arrow exists,making the diagram commute, if either i or p is a weak equivalence.
CM5 Every map $f : X \to Y$ has factorizations $f = p \cdot i$ and $f = q \cdot j$, in which i is a cofibration and a weak equivalence and p is a fibration, and j is a cofibration and q is a fibration and a weak equivalence.

There are various common adjectives which decorate closed model structures. For example, one says that the model structure on $\mathbf{M}$ is *simplicial* if the category can be enriched in simplicial sets in a way that behaves well with respect to cofibrations and fibrations, and the model structure is *proper* if weak equivalences are preserved by pullback along fibrations and pushout along cofibrations. Much more detail can be found in [24] or [29].

Theorem 1.13. *With these definitions given above of weak equivalence, cofibration and fibration, the category* $s\mathbf{Set}$ *of simplicial sets satisfies the axioms for a closed model category.*

Here are the basic ingredients of the proof:

Lemma 1.14. *A map* $p : X \to Y$ *is a trivial fibration if and only if it has the right lifting property with respect to all inclusions* $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.

The proof of Lemma 1.14 is formal. If p has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$ then it is a homotopy equivalence. Conversely, p has a factorization $p = q \cdot j$, where j is a cofibration and q has the right lifting property with respect to all maps $\partial\Delta^n \subset \Delta^n$, so that j is a trivial cofibration, and then p is a retract of q by a standard argument.

The following can be proved with simplicial approximation techniques [40].

Lemma 1.15. *Suppose that a simplicial set* X *has at most countably many non-degenerate simplices. Then the set of path components* $\pi_0|X|$ *and all homotopy groups* $\pi_n(|X|, x)$ *are countable.*

The following *bounded cofibration lemma* is a consequence.

Lemma 1.16. *Suppose that* $i : X \to Y$ *is a trivial cofibration and that* $A \subset Y$ *is a countable subcomplex. Then there is a countable subcomplex* $B \subset Y$ *with* $A \subset B$ *such that the map* $B \cap X \to B$ *is a trivial cofibration.*

Lemma 1.16 implies that the set of countable trivial cofibrations generate the class of all trivial cofibrations, while Lemma 1.14 implies that the set of all inclusions $\partial\Delta^n \subset \Delta^n$ generates the class of all cofibrations. Theorem 1.13 then follows from small object arguments.

A *Kan fibration* is a map $p : X \to Y$ of simplicial sets which has the right lifting property with respect to all inclusions $\Lambda^n_k \subset \Delta^n$. A *Kan complex* is a simplicial set X for which the canonical map $X \to *$ is a Kan fibration.

Remark 1.17. Every fibration is a Kan fibration. Every fibrant simplicial set is a Kan complex.

Kan complexes Y have combinatorially defined homotopy groups: if $x \in Y_0$ is a vertex of Y, then

$$\pi_n(Y, x) = \pi((\Delta^n, \partial\Delta^n), (Y, x))$$

where $\pi(\ ,\ )$ denotes simplicial homotopy classes of maps, of pairs in this case. The path components of any simplicial set X are defined by the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X,$$

where the maps $X_1 \to X_0$ are the face maps d_0, d_1. Say that a map $f : Y \to Y'$ of Kan complexes is a *combinatorial weak equivalence* if it induces isomorphisms

$$\pi_n(Y,x) \xrightarrow{\cong} \pi_n(Y', f(x))$$

for all $x \in Y_0$, and

$$\pi_0(Y) \xrightarrow{\cong} \pi_0(Y').$$

Going further requires the following major theorem, due to Quillen [58],[24]:

Theorem 1.18. *The realization of a Kan fibration is a Serre fibration.*

The proof of this result requires much of the classical homotopy theory of Kan complexes (in particular the theory of minimal fibrations), and will not be discussed here — see [24].

Here are the consequences:

Theorem 1.19 (Milnor theorem). *Suppose that Y is a Kan complex and that $\eta : Y \to S(|Y|)$ is the adjunction homomorphism. Then η is a combinatorial weak equivalence.*

It follows that the combinatorial homotopy groups of $\pi_n(Y,x)$ coincide up to natural isomorphism with the ordinary homotopy groups $\pi_n(|Y|,x)$ of the realization, for all Kan complexes Y. The proof of Theorem 1.19 is a long exact sequence argument, based on the path-loop fibre sequences in simplicial sets. These are Kan fibre sequences, and the key is to know, from Theorem 1.18 and Lemma 1.12, that their realizations are fibre sequences.

Theorem 1.20. *Every Kan fibration is a fibration.*

Proof (Sketch). The key step in the proof is to show, using Theorem 1.19, that every map $p : X \to Y$ which is a Kan fibration and a weak equivalence has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$. This is true if Y is a Kan complex, since p is then a combinatorial weak equivalence by Theorem 1.19. Maps which are weak equivalences and Kan fibrations are stable under pullback by Theorem 1.18 and Lemma 1.12. It follows from Theorem 1.19 that all fibres of the Kan fibration p are contractible. It also follows, by taking suitable pullbacks, that it suffices to assume that p has the form $p : X \to \Delta^k$. If F is the fibre of p over the vertex 0, then the Kan fibration p is fibrewise homotopy equivalent to the projection $F \times \Delta^k \to \Delta^k$ [24, I.10.6]. This projection has the desired right lifting property, as does any other Kan fibration in its fibre homotopy equivalence class — see [24, I.7.10]. $\square$

Remark 1.21. Theorem 1.20 implies that the model structure of Theorem 1.13 consists of cofibrations, Kan fibrations and weak equivalences. This is the standard, classical model structure for simplicial sets. The identification of the fibrations with Kan fibrations is the interesting part of this line of argument.

The realization functor preserves cofibrations and weak equivalences, and it follows that the adjoint pair

$$|\ | : s\mathbf{Set} \leftrightarrows \mathbf{CGHaus} : S,$$

is a *Quillen adjunction*. The following is a consequence of Theorem 1.19:

Theorem 1.22. *The adjunction maps $\eta : X \to S(|X|)$ and $\varepsilon : |S(Y)| \to Y$ are weak equivalences, for all simplicial sets X and spaces Y, respectively.*

In particular, the standard model structures on $s\mathbf{Set}$ and $\mathbf{CGHaus}$ are Quillen equivalent.

1.4 Projective model structure for diagrams

Suppose that I is a small category, and let $s\mathbf{Set}^I$ denote the category of I-diagrams of simplicial sets. The objects of this category are the functors $X : I \to s\mathbf{Set}$, and the morphisms $f : X \to Y$ are the natural transformations of functors. One often says that the category $s\mathbf{Set}^I$ is the *I-diagram category*.

There is a model structure on the I-diagram category, which was originally introduced by Bousfield and Kan [8], and for which the fibrations and weak equivalences are defined sectionwise. This model structure is now called the *projective model structure* on the I-diagram category. Cofibrant replacements in this structure are like projective resolutions of ordinary chain complexes.

Explicitly, a weak equivalence for this category is a map $f : X \to Y$ such that the simplicial set maps $f : X(i) \to Y(i)$ (the components of the natural transformation) are weak equivalences of simplicial sets for all objects i of I. One commonly says that such a map is a *sectionwise weak equivalence*. A map $p : X \to Y$ is said to be a *sectionwise fibration* if all components $p : X(i) \to Y(i)$ are fibrations of simplicial sets. Finally, a *projective cofibration* is a map which has the left lifting property with respect to all maps which are sectionwise weak equivalences and sectionwise fibrations, or which are sectionwise trivial fibrations.

The function complex $\mathbf{hom}(X,Y)$ for I-diagrams X and Y is the simplicial set whose n-simplices are all maps $X \times \Delta^n \to Y$ of I-diagrams. Here Δ^n has been identified with the constant I diagram which takes a morphism $i \to j$ to the identity map on Δ^n.

Observe that the i-sections functor $X \mapsto X(i)$ has a left adjoint

$$L_i : s\mathbf{Set} \to s\mathbf{Set}^I,$$

which is defined for simplicial sets K by

$$L_i(K) = \hom(i,\ ) \times K,$$

where $\hom(i,\ ) : I \to \mathbf{Set}$ is the functor which is represented by i.

Then we have the following:

Proposition 1.23. *The I-diagram category $s\mathbf{Set}^I$, together with the classes of projective cofibrations, sectionwise weak equivalences and sectionwise fibrations defined above, satisfies the axioms for a proper closed simplicial model category.*

Proof. A map $p : X \to Y$ is a sectionwise fibration if and only if it has the right lifting property with respect to all maps $L_i(\Lambda^n_k) \to L_i(\Delta^n)$ which are induced by inclusions of horns in simplices. A map $q : Z \to W$ of I-diagrams is a sectionwise fibration and a sectionwise weak equivalence if and only if it has the right lifting property with respect to all maps $L_i(\partial\Delta^n) \to L_i(\Delta^n)$.

Observe that every cofibration (monomorphism) $j : A \to B$ of simplicial sets induces a projective cofibration $j_* : L_i(A) \to L_i(B)$ of I-diagrams, and that this map j_* is a sectionwise cofibration. Observe also that if j is a trivial cofibration then j_* is a sectionwise weak equivalence.

It follows, by a standard small object argument, that every map $f : X \to Y$ of I-diagrams has factorizations

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd} \tag{1.3}$$

where i is a projective cofibration and a sectionwise weak equivalence and p is a sectionwise fibration, and j is a projective cofibration and q is a sectionwise trivial fibration. We have therefore proved the factorization axiom **CM5** for this structure.

The maps i and j in the diagram (1.3) are also sectionwise cofibrations, by construction, and the map i has the left lifting property with respect to all sectionwise fibrations.

In particular, if $\alpha : A \to B$ is a projective cofibration and a sectionwise weak equivalence, then α has a factorization

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "\alpha"'] & C \arrow[d, "p"] \\ & B \end{tikzcd}$$

where i is a projective cofibration, a sectionwise weak equivalence, and has the left lifting property with respect to all sectionwise fibrations, and p is a sectionwise fibration. The map p is also a sectionwise weak equivalence so the lift exists in the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"'] & C \arrow[d, "p"] \\ B \arrow[r, "1"'] \arrow[ur, dotted] & B \end{tikzcd}$$

It follows that α is a retract of the map i, and therefore has the left lifting property with respect to all projective fibrations. This proves the axiom **CM4**.

All of the other closed model axioms are easily verified.

Suppose that $j : K \to L$ is a cofibration of simplicial sets. The collection of all sectionwise cofibrations $\alpha : A \to B$ such that the induced map

$$(\alpha, j) : (B \times K) \cup (A \times L) \to B \times L$$

is a projective cofibration, is closed under pushout, composition, filtered colimits, retraction, and contains all maps $L_i M \to L_i N$ which are induced by cofibrations $M \to N$ of simplicial sets. This class of cofibrations α therefore contains all projective cofibrations. Observe further that the map (α, j) is a sectionwise weak equivalence if either α is a sectionwise equivalence or j is a weak equivalence of simplicial sets.

The I-diagram category therefore has a simplicial model structure in the sense that if $\alpha : A \to B$ is a projective cofibration and $j : K \to L$ is a cofibration of simplicial sets, then the map (α, j) is a projective cofibration, which is a sectionwise weak equivalence if either α is a sectionwise weak equivalence or j is a weak equivalence of simplicial sets.

All projective cofibrations are sectionwise cofibrations. Properness for the I-diagram category therefore follows from properness for simplicial sets.

The model structure for the I-diagram category $s\mathbf{Set}^I$ is *cofibrantly generated* , in the sense that the classes of projective cofibrations, respectively trivial projective cofibrations, are generated by the set of maps

$$L_i(\partial\Delta^n) \to L_i(\Delta^n), \tag{1.4}$$

respectively

$$L_i(\Lambda^n_k) \to L_i(\Delta^n). \tag{1.5}$$

This means that the cofibrations form the smallest class of maps which contains the set (1.4) and is closed under pushout, composition, filtered colimit and retraction. The class of trivial projective cofibrations is similarly the smallest class containing the set of maps (1.5) which has the same closure properties. The claim about the cofibrant generation is an artifact of the proof of Proposition 1.23.

The category $s\mathbf{Set}$ of simplicial sets is also cofibrantly generated. The simplicial set category is a category of I-diagrams, where I is the category with one morphism.

Chapter 2
Some topos theory

2.1 Grothendieck topologies

For our purposes, a *Grothendieck site* is a small category $\mathscr{C}$ equipped with a topology $\mathscr{T}$. A *Grothendieck topology* $\mathscr{T}$ consists of a collection of subfunctors

$$R \subset \hom(\ ,U), \quad U \in \mathscr{C},$$

called *covering sieves*, such that the following axioms hold:

1) (*base change*) If $R \subset \hom(\ ,U)$ is covering and $\phi : V \to U$ is a morphism of $\mathscr{C}$, then the subfunctor

$$\phi^{-1}(R) = \{\gamma : W \to V \mid \phi \cdot \gamma \in R\}$$

is covering for V.
2) (*local character*) Suppose that $R, R' \subset \hom(\ ,U)$ are subfunctors and R is covering. If $\phi^{-1}(R')$ is covering for all $\phi : V \to U$ in R, then R' is covering.
3) $\hom(\ ,U)$ is covering for all $U \in \mathscr{C}$

Typically, Grothendieck topologies arise from covering families in sites $\mathscr{C}$ having pullbacks. Covering families are sets of functors which generate covering sieves.

More explicitly, suppose that the category $\mathscr{C}$ has pullbacks. Since $\mathscr{C}$ is small, a *pretopology* (equivalently, a topology) $\mathscr{T}$ on $\mathscr{C}$ consists of families of sets of morphisms

$$\{\phi_\alpha : U_\alpha \to U\}, \quad U \in \mathscr{C},$$

called *covering families* , such that the following axioms hold:

1) Suppose that $\phi_\alpha : U_\alpha \to U$ is a covering family and that $\psi : V \to U$ is a morphism of $\mathscr{C}$. Then the collection $V \times_U U_\alpha \to V$ is a covering family for V.
2) If $\{\phi_\alpha : U_\alpha \to V\}$ is covering, and $\{\gamma_{\alpha,\beta} : W_{\alpha,\beta} \to U_\alpha\}$ is covering for all α, then the family of composites

$$W_{\alpha,\beta} \xrightarrow{\gamma_{\alpha,\beta}} U_\alpha \xrightarrow{\phi_\alpha} U$$

is covering.
3) The family $\{1 : U \to U\}$ is covering for all $U \in \mathscr{C}$.

Example 2.1. Let X be a topological space. The site $\mathrm{op}|_X$ is the poset of open subsets $U \subset X$. A covering family for an open subset U is an open cover $V_\alpha \subset U$.

Example 2.2. Suppose that X is a topological space. The site $\mathrm{loc}|_X$ is the category of all maps $f : Y \to X$ which are local homeomorphisms . Here, a map f is a local homeomorphism if each $x \in Y$ has a neighbourhood U such that $f(U)$ is open in X and the restricted map $U \to f(U)$ is a homeomorphism. A morphism of $\mathrm{loc}|_X$ is a commutative diagram

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

where f and f' are local homeomorphisms. A family $\{\phi_\alpha : Y_\alpha \to Y\}$ of local homeomorphisms (over X) is covering if $X = \cup \phi_\alpha(Y_\alpha)$.

Example 2.3. Suppose that S is a scheme (which is a topological space with sheaf of rings locally isomorphic to affine schemes $\mathrm{Sp}(R)$). The underlying topology on X is the Zariski topology. The *Zariski site* $Zar|_S$ is the poset with objects all open subschemes $U \subset S$. A family $V_\alpha \subset U$ is covering if $\cup V_\alpha = U$ as sets.

A scheme homomorphism $\phi : Y \to X$ is *étale at* $y \in Y$ if

a) $\mathscr{O}_y$ is a flat $\mathscr{O}_{f(y)}$-module (ϕ is flat at y).
b) ϕ is unramified at y: $\mathscr{O}_y / \mathscr{M}_{f(y)} \mathscr{O}_y$ is a finite separable field extension of $k(f(y))$.

Say that a map $\phi : Y \to X$ is *étale* if it is étale at every $y \in Y$ and locally of finite type (see [53], for example).

Example 2.4. Let S be a scheme scheme. The *étale site* $et|_S$ has as objects all étale maps $\phi : V \to S$ and all diagrams

$$\begin{tikzcd} V \arrow[rr] \arrow[dr, "\phi"'] & & V' \arrow[dl, "\phi'"] \\ & S & \end{tikzcd}$$

for morphisms (with ϕ, ϕ' étale). A covering family for the étale topology is a collection of étale morphisms $\phi_\alpha : V_\alpha \to V$ such that $V = \cup \phi_\alpha(V_\alpha)$ as a set. Equivalently every morphism $\mathrm{Sp}(\Omega) \to V$ lifts to some V_α if Ω is a separably closed field.

Example 2.5. The *Nisnevich site* $Nis|_S$ has the same underlying category as the étale site, namely all étale maps $V \to S$ and morphisms between them. A Nisnevich cover is a family of étale maps $V_\alpha \to V$ such that every morphism $\mathrm{Sp}(K) \to V$ lifts to some V_α where K is any field. Nisnevich originally called this topology the "completely decomposed topology" or "*cd*-topology" [56], because of the way it behaves over fields — see [37].

Example 2.6. A flat covering family of a scheme S is a set of flat morphisms $\phi_\alpha : S_\alpha \to S$ (ie. mophisms which are flat at each point) such that $S = \cup\phi_\alpha(S_\alpha)$ as a set (equivalently $\sqcup S_\alpha \to S$ is faithfully flat). The category $(Sch|_S)_{fl}$ is the "big" flat site . Pick a large cardinal κ; then $(Sch|_S)$ is the category of S-schemes $X \to S$ such that the cardinality of both the underlying point set of X and all sections $\mathscr{O}_X(U)$ of its sheaf of rings are bounded above by κ.

Example 2.7. There are corresponding big sites $(Sch|_S)_{Zar}$, $(Sch|_S)_{et}$, $(Sch|_S)_{Nis}$, and one can play similar games with topological spaces.

Example 2.8. Suppose that $G = \{G_i\}$ is profinite group such that all $G_j \to G_i$ are surjective group homomorphisms. Write also $G = \varprojlim G_i$. A *discrete G-set* is a set X with G-action which factors through an action of G_i for some i. Write $G-\mathbf{Set}_{df}$ for the category of G-sets which are both discrete and finite. A family $U_\alpha \to X$ in this category is covering if and only if the induced map $\bigsqcup U_\alpha \to X$ is surjective.

Example 2.9. Suppose that $\mathscr{C}$ is any small category. Say that $R \subset \hom(\ ,x)$ is covering if and only if $1_x \in R$. This is the *chaotic topology* on $\mathscr{C}$.

Example 2.10. Suppose that $\mathscr{C}$ is a site and that $U \in \mathscr{C}$. Then the *slice category* $\mathscr{C}/U$ has for objects all morphisms $V \to U$ of $\mathscr{C}$, and its morphisms are the commutative triangles

$$\begin{tikzcd} V \arrow[dd] \arrow[dr] & \\ & U \\ V' \arrow[ur] & \end{tikzcd}$$

The category $\mathscr{C}/U$ inherits a topology from $\mathscr{C}$: a collection of maps $V_\alpha \to V \to U$ is covering if and only if the family $V_\alpha \to V$ covers V.

A *presheaf* (of sets) on a Grothendieck site $\mathscr{C}$ is a set-valued contravariant functor defined on $\mathscr{C}$, or equivalently a functor $\mathscr{C}^{op} \to \mathbf{Set}$ defined on the opposite category $\mathscr{C}^{op}$, where $\mathscr{C}^{op}$ is the category $\mathscr{C}$ with its arrows reversed. The presheaves on $\mathscr{C}$ form a category whose morphisms are natural transformation, which is often denoted by $\mathbf{Pre}(\mathscr{C})$ and is called the presheaf category for the site $\mathscr{C}$.

One can talk about presheaves taking values in any category $\mathbf{E}$, and following [49] (for example) we can write $\mathbf{Pre}(\mathscr{C},\mathbf{E})$ for the corresponding category of $\mathbf{E}$-presheaves on $\mathscr{C}$, or functors $\mathscr{C}^{op} \to \mathbf{E}$ and their natural transformations. The shorter notation

$$s\mathbf{Pre}(\mathscr{C}) := \mathbf{Pre}(\mathscr{C}, s\mathbf{Set})$$

denotes the category of presheaves $\mathscr{C}^{op} \to s\mathbf{Set}$ on $\mathscr{C}$ taking values in simplicial sets — this is the category of *simplicial presheaves* on $\mathscr{C}$. One views simplicial presheaves as either simplicial objects in presheaves, or as presheaves in simplicial sets.

A *sheaf* (of sets) on $\mathscr{C}$ is a presheaf $F : \mathscr{C}^{op} \to \mathbf{Set}$ such that the canonical map

$$F(U) \to \varprojlim_{V \to U \in R} F(V) \tag{2.1}$$

is an isomorphism for each covering sieve $R \subset \hom(\ ,U)$. Morphisms of sheaves are natural transformations; write $\mathbf{Shv}(\mathscr{C})$ for the corresponding category. The *sheaf category* $\mathbf{Shv}(\mathscr{C})$ is a full subcategory of $\mathbf{Pre}(\mathscr{C})$.

There is an analogous definition for sheaves in any complete category $\mathbf{E}$, and one would write $\mathbf{Shv}(\mathscr{C},\mathbf{E})$ for the corresponding category. The assertion that the category $\mathbf{E}$ is complete means that it has all small limits, so requiring that the morphism (2.1) should be an isomorphism for the functor $F : \mathscr{C}^{op} \to \mathbf{E}$ to be a sheaf makes sense.

Following the convention for simplicial presheaves displayed above, use the notation

$$s\mathbf{Shv}(\mathscr{C}) := \mathbf{Shv}_{\mathscr{C}}(\mathscr{C}, s\mathbf{Set})$$

for the category of *simplicial sheaves* on the site $\mathscr{C}$.

Exercise 2.11. If the topology on $\mathscr{C}$ is defined by a pretopology (so that $\mathscr{C}$ has all pullbacks), show that F is a sheaf if and only if all diagrams

$$F(U) \to \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \times_U U_\beta)$$

arising from covering families $U_\alpha \to U$ are equalizers.

Lemma 2.12. *1) If $R \subset R' \subset \hom(\ ,U)$ and R is covering then R' is covering.*
2) If $R,R' \subset \hom(\ ,U)$ are covering then $R \cap R'$ is covering.
3) Suppose that $R \subset \hom(\ ,U)$ covering and that $S_\phi \subset \hom(\ ,V)$ is covering for all $\phi : V \to U$ of R. Let $R * S$ be the sieve which is generated by the composites

$$W \xrightarrow{\gamma} V \xrightarrow{\phi} U$$

with $\phi \in R$ and $\gamma \in S_\phi$. Then $R * S$ is covering.

Proof. For 1), one shows that $\phi^{-1}(R) = \phi^{-1}(R')$ for all $\phi \in R$, so that R' is covering by the local character axiom. The relation $\phi^{-1}(R \cap R') = \phi^{-1}(R')$ for all $\phi \in R$ implies that $R \cap R'$ is covering, giving 2). Statement 3) is proved by observing that $S_\phi \subset \phi^{-1}(R * S)$ for all $\phi \in R$. □

Suppose that $R \subset \hom(\ ,U)$ is a sieve, and F is a presheaf on $\mathscr{C}$. Write

$$F(U)_R = \varprojlim_{V \xrightarrow{\phi} U \in R} F(V)$$

Write $\{x_\phi\}$ to denote elements of $F(U)_R$, and call them *R-compatible families* in F.

If $S \subset R$ then there is an obvious map

$$F(U)_R \to F(U)_S$$

Write

$$LF(U) = \varinjlim_{R} F(U)_R$$

where the colimit is indexed over all covering sieves $R \subset \hom(\ ,U)$. This colimit is filtered by Lemma 2.12. Elements of $LF(U)$ are classes $[\{x_\phi\}]$ of compatible families. Then the assignment $U \mapsto LF(U)$ defines a presheaf, and there is a natural presheaf map

$$\nu : F \to LF$$

Say that a presheaf G is *separated* if (equivalently)

1) the map $\nu : G \to LG$ is a monomorphism in each section, ie. all functions $G(U) \to LG(U)$ are injective, or
2) Given $x,y \in G(U)$, if there is a covering sieve $R \subset \hom(\ ,U)$ such that $\phi^*(x) = \phi^*(y)$ for all $\phi \in R$, then $x = y$.

Lemma 2.13. *1) The presheaf LF is separated, for all presheaves F.*
2) If G is a separated presheaf then LG is a sheaf.
3) If $f : F \to G$ is a presheaf map and G is a sheaf, then f factors uniquely through a presheaf map $f_* : LF \to G$.

It follows from Lemma 2.13 that the object L^2F is a sheaf for every presheaf F, and the functor $F \mapsto L^2F$ is left adjoint to the inclusion $\mathbf{Shv}(\mathscr{C}) \subset \mathbf{Pre}(\mathscr{C})$. The unit of the adjunction is the composite

$$F \xrightarrow{\nu} LF \xrightarrow{\nu} L^2F \tag{2.2}$$

One often writes $\tilde{F} := L^2F$ for the sheaf associated to the presheaf F, and in these terms it is standard to write $\eta : F \to \tilde{F}$ for the composite (2.2).

Proof (of Lemma 2.13). Suppose that $\psi^*(x) = \psi^*(y)$ for all $\psi : W \to U$ is some covering sieve $S \subset \hom(\ ,U)$, where $x,y \in LF(U)$. We can assume that $x = [\{x_\phi\}]$ and $y = [\{y_\phi\}]$ are represented by compatible families defined on the same covering sieve $R \subset \hom(\ ,U)$. By restricting to the intersection $S \cap R$ (Lemma 2.12), we can also assume that $S = R$. It follows that, for each $\phi : V \to U$ in R, there is a covering sieve T_ϕ such that

$$x_{\phi\gamma} = \gamma^*(x_\phi) = \gamma^*(y_\phi) = y_{\phi\gamma}$$

for each $\gamma : W \to V$ in T_ϕ. The compatible families $\{x_\phi\}$ and $\{y_\phi\}$ therefore restrict to the same compatible family on the covering sieve $R * T \subset R$, so that $[\{x_\phi\}] = [\{y_\phi\}]$. The presheaf LF is therefore separated, giving statement 1).

If $\phi : V \to U$ is a member of a covering sieve $R \subset \hom(\ ,U)$, then $\phi^{-1}(R) = \hom(\ ,V)$ is the unique covering sieve for V which contains the identity $1_V : V \to V$. It follows that if the compatible family $\{x_\phi\}$, $\phi : V \to U$ in R, is an R-compatible family, then $\phi^*[\{x_\phi\}] = \nu(x_\phi)$ for all $\phi \in R$.

Suppose that G is separated, and that $[\{v_\phi\}] \in LG(U)_R$ is an R-compatible family. Then each v_ϕ lifts locally to G along some covering sieve T_ϕ according to the previous paragraph, so there is a refinement $R * T \subset R$ of covering sieves such that $v_\psi = \nu(x_\psi)$ for each $\psi : W \to U$ of $R * T$. The presheaf G is separated, so that the elements x_ψ define an element of $G(U)_{R*T}$ and an element $[\{x_\psi\}]$ of $LG(U)$. Then

$\phi^*[\{x_\psi\}] = v_\phi$ for each $\phi \in R$ since LG is separated, and it follows that the canonical function

$$LG(U) \to LG(U)_R$$

is surjective. This function is injective since LG is separated. Thus, LG is a sheaf, giving statement 2).

If G is a sheaf, then the presheaf map $\nu : G \to LG$ is an isomorphism essentially by definition, and statement 3) follows. □

2.2 Exactness properties

Lemma 2.14. *1) The associated sheaf functor preserves all finite limits.*
2) The sheaf category $\mathbf{Shv}(\mathscr{C})$ *is complete and co-complete. Limits are formed sectionwise.*
3) Every monomorphism in $\mathbf{Shv}(\mathscr{C})$ *is an equalizer.*
4) If the sheaf morphism $\theta : F \to G$ *is both a monomorphism and an epimorphism, then* θ *is an isomorphism.*

Proof. Statement 1) is proved by observing that LF is defined by filtered colimits, and finite limits commute with filtered colimits.

If $X : I \to \mathbf{Shv}(\mathscr{C})$ is a diagram of sheaves, then the colimit in the sheaf category is $L^2(\varinjlim X)$, where $\varinjlim X$ is the presheaf colimit, giving statement 2).

If $A \subset X$ is a subset, then there is an equalizer

$$A \longrightarrow X \underset{*}{\overset{p}{\rightrightarrows}} X/A$$

The same holds for subobjects $A \subset X$ of presheaves, and hence for subobjects of sheaves, since the associated sheaf functor L^2 preserves finite limits. Statement 3) follows.

For statement 4), observe that the map θ appears in an equalizer

$$F \xrightarrow{\theta} G \underset{g}{\overset{f}{\rightrightarrows}} K$$

since θ is a monomorphism. But θ is an epimorphism, so $f = g$. But then $1_G : G \to G$ factors through θ, giving a section $\sigma : G \to F$. Finally, $\theta\sigma\theta = \theta$ and θ is a monomorphism, so $\sigma\theta = 1$. □

Here are some fundamental definitions:

1) A presheaf map $f : F \to G$ is a *local epimorphism* if for each $x \in G(U)$ there is a covering $R \subset \hom(\ ,U)$ such that $\phi^*(x) = f(y_\phi)$ for some y_ϕ, for all $\phi \in R$.
2) $f : F \to G$ is a *local monomorphism* if given $x, y \in F(U)$ such that $f(x) = f(y)$, there is a covering $R \subset \hom(\ ,U)$ such that $\phi^*(x) = \phi^*(y)$ for all $\phi \in R$.

3) A presheaf map $f : F \to G$ which is both a local epimorphism and a local monomorphism is a *local isomorphism* .

Example 2.15. The canonical map $\nu : F \to LF$ is a local isomorphism for all presheaves F. The fact that ν is a local monomorphism is a consequence of the definitions, and the claim that ν is a local epimorphism appears as a detail in the proof of Lemma 2.13

It follows that the associated sheaf map $\eta : F \to L^2F$ is also a local isomorphism, for all presheaves F.

Lemma 2.16. *Suppose that $f : F \to G$ is a presheaf morphism. Then f induces an isomorphism (respectively monomorphism, epimorphism) $f_* : L^2F \to L^2G$ of associated sheaves if and only if f is a local isomorphism (respectively local monomorphism, local epimorphism) of presheaves.*

Proof. It is an exercise to show that, given a commutative diagram

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[dr, "h"'] & F' \arrow[d, "f"] \\ & F'' \end{tikzcd}$$

of presheaf morphisms, if any two of f, g and h are local isomorphisms, then so is the third. It is an exercise to show that a sheaf map $g : E \to E'$ is a monomorphism (respectively epimorphism) if and only if it is a local monomorphism (respectively local epimorphism). Now use the comparison diagram

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "f"'] & L^2F \arrow[d, "f_*"] \\ G \arrow[r, "\eta"'] & L^2G \end{tikzcd}$$

to finish the proof of the Lemma. □

A *Grothendieck topos* is a category $\mathscr{E}$ which is equivalent to a sheaf category $\mathbf{Shv}(\mathscr{C})$ on some Grothendieck site $\mathscr{C}$.

Grothendieck toposes are characterized by exactness properties :

Theorem 2.17 (Giraud). *A category $\mathscr{E}$ having all finite limits is a Grothendieck topos if and only if it has the following properties:*

1) *The category $\mathscr{E}$ has all small coproducts; they are disjoint and stable under pullback.*
2) *Every epimorphism of $\mathscr{E}$ is a coequalizer.*
3) *Every equivalence relation $R \rightrightarrows E$ in $\mathscr{E}$ is a kernel pair and has a quotient.*
4) *Every coequalizer $R \rightrightarrows E \to Q$ is stably exact.*

5) There is a set of objects which generates the category $\mathscr{E}$.

A sketch proof of Giraud's Theorem appears below, but the result is proved in many places — see, for example, [52], [59].

Here are the definitions of the terms appearing in the statement of Theorem 2.17:

1) The coproduct $\bigsqcup_i A_i$ is *disjoint* if all diagrams

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A_j \arrow[d] \\ A_i \arrow[r] & \bigsqcup_i A_i \end{tikzcd}$$

are pullbacks for $i \neq j$. The coproduct $\bigsqcup_i A_i$ is *stable under pullback* if all diagrams

$$\begin{tikzcd} \bigsqcup_i C \times_B A_i \arrow[r] \arrow[d] & \bigsqcup_i A_i \arrow[d] \\ C \arrow[r] & B \end{tikzcd}$$

are pullbacks.

3) An *equivalence relation* is a monomorphism $m = (m_0, m_1) : R \to E \times E$ such that

a) the diagonal $\Delta : E \to E \times E$ factors through m (ie. $a \sim a$)
b) the composite $R \xrightarrow{m} E \times E \xrightarrow{\tau} E \times E$ factors through m (ie. $a \sim b \Rightarrow b \sim a$).
c) the map

$$(m_0 m_{0*}, m_1 m_{1*}) : R \times_E R \to E \times E$$

factors through m (this is transitivity) where the pullback is defined by

$$\begin{tikzcd} R \times_E R \arrow[r, "m_{1*}"] \arrow[d, "m_{0*}"'] & R \arrow[d, "m_0"] \\ R \arrow[r, "m_1"'] & E \end{tikzcd}$$

The *kernel pair* of a morphism $u : E \to D$ is a pullback

$$\begin{tikzcd} R \arrow[r, "m_1"] \arrow[d, "m_0"'] & E \arrow[d, "u"] \\ E \arrow[r, "u"'] & D \end{tikzcd}$$

(Exercise: show that every kernel pair is an equivalence relation).

A *quotient* for an equivalence relation $(m_0, m_1) : R \to E \times E$ is a coequalizer

$$R \overset{m_0}{\underset{m_1}{\rightrightarrows}} E \longrightarrow E/R$$

4) A coequalizer $R \rightrightarrows E \to Q$ is *stably exact* if the induced diagram

$$R \times_Q Q' \rightrightarrows E \times_Q Q' \to Q'$$

is a coequalizer for all morphisms $Q' \to Q$.

5) Following [6], a *generating set* is a set of objects S which detects the difference between maps. This means precisely that the map

$$\bigsqcup_{x \to E} x \to E$$

which is defined by all maps $x \to E$ with $x \in S$, is an epimorphism, and this for all objects E of $\mathscr{E}$.

Exercise 2.18. Show that any category $\mathbf{Shv}(\mathscr{C})$ on a site $\mathscr{C}$ satisfies the conditions of Giraud's theorem. The family $L^2 \hom(\ ,U)$, $U \in \mathscr{C}$ is a set of generators.

Proof (Sketch proof of Theorem 2.17). The key is to show that a category $\mathscr{E}$ which satisfies the conditions of the Theorem is cocomplete. In view of the assumption that $\mathscr{E}$ has all small coproducts it is enough to show that $\mathscr{E}$ has all coequalizers. The coequalizer of the maps $f_1, f_2 : E' \to E$ is constructed by taking the canonical map $E \to E/R$, where R is the minimal equivalence relation which is contains (f_1, f_2) in the sense that there is a commutative diagram

$$\begin{tikzcd} & R \arrow[d] \\ E' \arrow[ur] \arrow[r, "(f_1,f_2)"'] & E \times E \end{tikzcd}$$

See also [52, p.575].

Suppose that S is the set of generators for $\mathscr{E}$ prescribed by Giraud's theorem, and let $\mathscr{C}$ be the full subcategory of $\mathscr{E}$ on the set of objects A. A subfunctor $R \subset \hom(\ ,x)$ on $\mathscr{C}$ is covering if the map

$$\bigsqcup_{y \to x \in R} y \to x$$

is an epimorphism of $\mathscr{E}$. In such cases, there is a coequalizer

$$\bigsqcup_{y_0 \to y_1 \to x} y_0 \rightrightarrows \bigsqcup_{y \to x \in R} y \to x$$

where the displayed strings $y_0 \to y_1 \to x$ are morphisms between generators such that $y_1 \to x$ is in R.

It follows that every object $E \in \mathscr{E}$ represents a sheaf $\hom(\ ,E)$ on $\mathscr{C}$, and a sheaf F on $\mathscr{C}$ determines an object

$$\varinjlim_{\hom(\ ,y)\to F} y$$

of $\mathscr{E}$.

The adjunction

$$\hom(\varinjlim_{\hom(\ ,y)\to F} y, E) \cong \hom(F, \hom(\ ,E))$$

determines an adjoint equivalence between $\mathscr{E}$ and $\mathbf{Shv}(\mathscr{C})$. □

The strategy of the proof of Giraud's Theorem is arguably as important as the statement of the Theorem itself. Here are some examples;

Example 2.19. Suppose that G is a sheaf of groups, and let $G-\mathbf{Shv}(\mathscr{C})$ denote the category of all sheaves X admitting G-action, with equivariant maps between them. $G-\mathbf{Shv}(\mathscr{C})$ is a Grothendieck topos, called the *classifying topos* for G, by Giraud's theorem. The objects $G \times L^2 \hom(\ ,x)$ form a generating set.

Example 2.20. If $G = \{G_i\}$ is a profinite group such that all transition maps $G_i \to G_j$ are surjective, then the category $G-\mathbf{Set}_d$ of discrete G-sets is a Grothendieck topos. A discrete G-set is a set X equipped with a pro-map $G \to \mathrm{Aut}(X)$. The finite discrete G-sets form a generating set for this topos, and the full subcategory on the finite discrete G-sets is the site prescribed by Giraud's theorem.

If the profinite group G is the absolute Galois group of a field k, then the category $G-\mathbf{Set}_d$ of discrete G-sets is equivalent to the category $\mathbf{Shv}(et|_k)$ of sheaves on the étale site for k. More generally, if S is a locally Noetherian connected scheme with geometric point x, and the profinite group $\pi_1(S,x)$ is the Grothendieck fundamental group, then the category of discrete $\pi_1(S,x)$-sets is equivalent to the category of sheaves on the finite étale site $fet|_S$ for the scheme S. See [1], [53].

2.3 Geometric morphisms

Suppose that $\mathscr{C}$ and $\mathscr{D}$ are Grothendieck sites. A *geometric morphism*

$$f : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Shv}(\mathscr{D})$$

consists of functors $f_* : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Shv}(\mathscr{D})$ and $f^* : \mathbf{Shv}(\mathscr{D}) \to \mathbf{Shv}(\mathscr{C})$ such that f^* is left adjoint to f_* and f^* preserves finite limits.

The left adjoint f^* is called the *inverse image* functor, while f_* is called the *direct image* . The inverse image functor f^* is left and right exact in the sense that it preserves all finite colimits and limits, respectively. The direct image functor f_* is usually not left exact (ie. it may not preserve finite colimits), and therefore has derived functors.

Example 2.21. Suppose $f : X \to Y$ is a continuous map of topological spaces. Pullback along f induces a functor $\mathrm{op}|_Y \to \mathrm{op}|_X$ which takes an open subset $U \subset Y$ to $f^{-1}(U)$. Open covers pull back to open covers, so that if F is a sheaf on X, then composition with the pullback gives a sheaf f_*F on Y with $f_*F(U) = F(f^{-1}(U))$. The resulting functor

$$f_* : \mathbf{Shv}(\mathrm{op}|_X) \to \mathbf{Shv}(\mathrm{op}|_Y)$$

is the direct image. It extends to a direct image functor

$$f_* : \mathbf{Pre}(\mathrm{op}|_X) \to \mathbf{Pre}(\mathrm{op}|_Y)$$

which is defined in the same way

The left Kan extension

$$f^p : \mathbf{Pre}(\mathrm{op}|_Y) \to \mathbf{Pre}(\mathrm{op}|_X)$$

of the presheaf-level direct image is defined by

$$f^p G(V) = \varinjlim G(U)$$

where the colimit is indexed over all diagrams

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in which the vertical maps are inclusions of open subsets. The category $\mathrm{op}|_Y$ has all products (ie. intersections), so the colimit is filtered. The functor $G \mapsto f^p G$ therefore commutes with finite limits. The sheaf theoretic inverse image functor

$$f^* : \mathbf{Shv}(\mathrm{op}|_Y) \to \mathbf{Shv}(\mathrm{op}|_X)$$

is defined by $f^*(G) = L^2 f^p(G)$. The resulting pair of functors forms a geometric morphism $f : \mathbf{Shv}(\mathrm{op}|_X) \to \mathbf{Shv}(\mathrm{op}|_Y)$.

Example 2.22. Suppose that $f : X \to Y$ is a morphism of schemes. Etale maps (respectively covers) are stable under pullback, and so there is a functor $\mathrm{et}|_Y \to \mathrm{et}|_X$ which is defined by pullback. If F is a sheaf on $\mathrm{et}|_X$ then there is a sheaf f_*F on $\mathrm{et}|_Y$ which is defined by $f_*F(V \to Y) = f(X \times_Y V \to X)$

The restriction functor $f_* : \mathbf{Pre}(\mathrm{et}|_X) \to \mathbf{Pre}(\mathrm{et}|_Y)$ has a left adjoint f^p defined by

$$f^p G(U \to X) = \varinjlim \ G(V)$$

where the colimit is indexed over the category of all diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where both vertical maps are étale. The colimit is filtered, because étale maps are stable under pullback and composition. The inverse image functor

$$f^* : \mathbf{Shv}(\mathrm{et}|_Y) \to \mathbf{Shv}(et|_X)$$

is defined by $f^*F = L^2 f^p F$, and so f induces a geometric morphism $f : \mathbf{Shv}(\mathrm{et}|_X) \to \mathbf{Shv}(\mathrm{et}|_Y)$.

A morphism of schemes $f : X \to Y$ induces a geometric morphism $f : \mathbf{Shv}(?|_X) \to \mathbf{Shv}(?|_Y)$ and/or $f : \mathbf{Shv}(Sch|_X)_? \to \mathbf{Shv}(Sch|_Y)_?$ for all of the geometric topologies (eg. Zariski, flat, Nisnevich, qfh, ...), by similar arguments.

Example 2.23. A *point* of $\mathbf{Shv}(\mathscr{C})$ is a geometric morphism $\mathbf{Set} \to \mathbf{Shv}(\mathscr{C})$. Every point $x \in X$ of a topological space X determines a continuous map $\{x\} \subset X$ and hence a geometric morphism

$$\mathbf{Set} \cong \mathbf{Shv}(\mathrm{op}|_{\{x\}}) \xrightarrow{x} \mathbf{Shv}(\mathrm{op}|_X)$$

The set

$$x^*F = \varinjlim_{x \in U} F(U)$$

is the *stalk* of F at x

Example 2.24. Suppose that k is a field. Any scheme homomorphism $x : \mathrm{Sp}(k) \to X$ induces a geometric morphism

$$\mathbf{Shv}(et|_k) \to \mathbf{Shv}(et|_X)$$

If the field k happens to be algebraically closed (or separably closed), then there is an equivalence $\mathbf{Shv}(et|_k) \simeq \mathbf{Set}$ and the resulting geometric morphism $x : \mathbf{Set} \to \mathbf{Shv}(\mathrm{et}|_X)$ is called a *geometric point* of X. The inverse image functor

$$F \mapsto x^*F = \varinjlim F(U)$$

is the stalk of the sheaf (or presheaf) F at x. The indicated colimit is indexed by the filtered category of diagrams

$$\begin{tikzcd} & U \arrow[d, "\phi"] \\ \mathrm{Sp}(k) \arrow[ur] \arrow[r, "x"'] & X \end{tikzcd}$$

in which the vertical maps ϕ are étale. These diagrams are the *étale neighbourhoods* of the geometric point x.

Example 2.25. Suppose that S and T are topologies on a site $\mathscr{C}$ and that $S \subset T$. In other words, T has more covers than S and hence refines S. Then every sheaf for T is a sheaf for S; write

$$\pi_* : \mathbf{Shv}_T(\mathscr{C}) \subset \mathbf{Shv}_S(\mathscr{C})$$

for the corresponding inclusion functor. The associated sheaf functor for the topology T gives a left adjoint π^* for the inclusion functor π_*, and the functor π^* preserves finite limits.

In particular, comparing an arbitrary topology with the chaotic topology on a site $\mathscr{C}$ gives a geometric morphism

$$\mathbf{Shv}(\mathscr{C}) \to \mathbf{Pre}(\mathscr{C})$$

for which the direct image is the inclusion of the sheaf category in the presheaf category, and the inverse image is the associated sheaf functor.

A *site morphism* is a functor $f : \mathscr{D} \to \mathscr{C}$ between Grothendieck sites such that

1) If F is a sheaf on $\mathscr{C}$, then the composite functor

$$\mathscr{D}^{op} \xrightarrow{f^{op}} \mathscr{C}^{op} \xrightarrow{F} \mathbf{Set}$$

is a sheaf on $\mathscr{D}$.
2) Suppose that f^p is the left adjoint of the functor

$$f_* : \mathbf{Pre}(\mathscr{C}) \to \mathbf{Pre}(\mathscr{D})$$

which is defined by precomposition with f^{op}. Then the functor f^p is left exact in the sense that it preserves all finite limits.

One often paraphrases the requirement 1) by saying that the functor f_* should be *continuous*: it restricts to a functor

$$f_* : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Shv}(\mathscr{D}).$$

The left adjoint

$$f^* : \mathbf{Shv}(\mathscr{D}) \to \mathbf{Shv}(\mathscr{C})$$

is defined for a sheaf E by $f^*(E) = L^2 f^p(E)$. The functor f^* preserves finite limits since the presheaf-level functor f^p is required to have this property. It follows that every site morphism $f : \mathscr{D} \to \mathscr{C}$ induces a geometric morphism

$$f : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Shv}(\mathscr{D}).$$

Suppose that $g : \mathscr{D} \to \mathscr{C}$ is a functor between Grothendieck sites such that

1′) If $R \subset \hom(\ ,U)$ is a covering sieve for $\mathscr{D}$ then the image $g(R)$ of the set of morphisms of R in $\mathscr{C}$ generates a covering sieve for $\mathscr{C}$.
2′) The sites $\mathscr{D}$ and $\mathscr{C}$ have all finite limits, and the functor g preserves them.

It is an exercise to show that such a functor g must satisfy the corresponding properties 1) and 2) above, and therefore defines a site morphism. The functor g is what Mac Lane and Moerdijk [52] would call a site morphism, while the definition in use here is consistent with that of SGA4 [2].

In many practical cases, such as Example 2.21 and Example 2.22 above, geometric morphism are induced by functors g which satisfy conditions 1′) and 2′).

2.4 Points

Say that a Grothendieck topos $\mathbf{Shv}(\mathscr{C})$ has *enough points* if there is a set of geometric morphisms $x_i : \mathbf{Set} \to \mathbf{Shv}(\mathscr{C})$ such that the induced functor

$$\mathbf{Shv}(\mathscr{C}) \xrightarrow{(x_i^*)} \prod_i \mathbf{Set}$$

is faithful.

Lemma 2.26. *Suppose that $f : \mathbf{Shv}(\mathscr{D}) \to \mathbf{Shv}(\mathscr{C})$ is a geometric morphism. Then the following are equivalent:*

a) The functor $f^* : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Shv}(\mathscr{D})$ is faithful.
b) The functor f^* reflects isomorphisms
c) The functor f^* reflects epimorphisms
d) The functor f^* reflects monomorphisms

Proof. Suppose that f^* is faithful, which means that $f^*(g_1) = f^*(g_2)$ implies that $g_1 = g_2$. Suppose that $m : F \to G$ is a morphism of $\mathbf{Shv}(\mathscr{C})$ such that $f^*(m)$ is a monomoprhism. If $m \cdot f_1 = m \cdot f_2$ then $f^*(f_1) = f^*(f_2)$ so $f_1 = f_2$. The map m is therefore a monomorphism. Similarly, the functor f^* reflects epimorphisms and hence isomorphisms.

Suppose that the functor f^* reflects epimorphisms and suppose given morphisms $g_1, g_2 : F \to G$ such that $f^*(g_1) = f^*(g_2)$. We have equality $g_1 = g_2$ if and only if their equalizer $e : E \to F$ is an epimorphism. But f^* preserves equalizers and reflects epimorphisms, so e is an epimorphism and $g_1 = g_2$. The other arguments are similar. □

Here are some basic definitions:

1) A *lattice* L is a partially ordered set which has all finite coproducts $x \vee y$ and all finite products $x \wedge y$.
2) A lattice L has 0 and 1 if it has an initial and terminal object, respectively.

3) A lattice L is said to be *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all x, y, z.
4) A *complement* for x in a lattice L with 0 and 1 is an element a such that $x \vee a = 1$ and $x \wedge a = 0$. If L is also distributive the complement, if it exists, is unique: if b is another complement for x, then

$$\begin{aligned} b &= b \wedge 1 = b \wedge (x \vee a) = (b \wedge x) \vee (b \wedge a) \\ &= (x \wedge a) \vee (b \wedge a) = (x \vee b) \wedge a = a \end{aligned}$$

One usually writes $\neg x$ for the complement of x.
5) A *Boolean algebra* $\mathscr{B}$ is a distributive lattice with 0 and 1 in which every element has a complement.
6) A lattice L is said to be *complete* if it has all small limits and colimits (aka. all small meets and joins).
7) A *frame* P is a lattice which has all small joins (and all finite meets) and which satisfies an infinite distributive law

$$U \wedge (\bigvee_i V_i) = \bigvee_i (U \wedge V_i)$$

Example 2.27. 1) The poset $\mathscr{O}(T)$ of open subsets of a topological space T is a frame. Every continuous map $f : S \to T$ induces a morphism of frames $f^{-1} : \mathscr{O}(T) \to \mathscr{O}(S)$, defined by $U \mapsto F^{-1}(U)$.
2) The power set $\mathscr{P}(I)$ of a set I is a complete Boolean algebra.
3) Every complete Boolean algebra $\mathscr{B}$ is a frame. In effect, every join is a filtered colimit of finite joins.

Every frame A has a canonical Grothendieck topology: a family $y_i \leq x$ is covering if $\bigvee_i y_i = x$. Write $\mathbf{Shv}(A)$ for the corresponding sheaf category. Every complete Boolean algebra $\mathscr{B}$ is a frame, and therefore has an associated sheaf category $\mathbf{Shv}(\mathscr{B})$.

Example 2.28. Suppose that I is a set. Then there is an equivalence

$$\mathbf{Shv}(\mathscr{P}(I)) \simeq \prod_{i \in I} \mathbf{Set}$$

Any set I of points $x_j : \mathbf{Set} \to \mathbf{Shv}(\mathscr{C}$ assembles to give a geometric morphism

$$x : \mathbf{Shv}(\mathscr{P}(I)) \to \mathbf{Shv}(\mathscr{C}).$$

Observe that the sheaf category $\mathbf{Shv}(\mathscr{C})$ has enough points if there is such a set I of points such that the inverse image functor x^* for the geometric morphism x is faithful.

Lemma 2.29. *Suppose that F is a sheaf of sets on a complete Boolean algebra $\mathscr{B}$. Then the poset $Sub(F)$ of subobjects of F is a complete Boolean algebra.*

Proof. The poset $Sub(F)$ is a frame, by an argument on the presheaf level. It remains to show that every object $G \in Sub(F)$ is complemented. The obvious candidate for $\neg G$ is

$$\neg G = \bigvee_{H \leq F,\ H \wedge G = \emptyset} H$$

and we need to show that $G \bigvee \neg G = F$.

Every $K \leq \hom(\ ,A)$ is representable: in effect,

$$K = \varinjlim_{\hom(\ ,B)\to K} \hom(\ ,B) = \hom(\ ,C)$$

where

$$C = \bigvee_{\hom(\ ,B)\to K} B \in \mathscr{B}.$$

It follows that $Sub(\hom(\ ,A)) \cong Sub(A)$ is a complete Boolean algebra.

Consider all diagrams

$$\begin{tikzcd}
\phi^{-1}(G) \arrow[r] \arrow[d] & G \arrow[d] \\
\hom(\ ,A) \arrow[r, "\phi"'] & F
\end{tikzcd}$$

There is an induced pullback

$$\begin{tikzcd}
\phi^{-1}(G) \vee \neg\phi^{-1}(G) \arrow[r] \arrow[d, "\cong"'] & G \vee \neg G \arrow[d] \\
\hom(\ ,A) \arrow[r, "\phi"'] & F
\end{tikzcd}$$

The sheaf F is a union of its representable subsheaves, since all ϕ are monomorphisms since all $\hom(\ ,A)$ are subobjects of the terminal sheaf. It follows that $G \vee \neg G = F$. □

Lemma 2.30. *Suppose that $\mathscr{B}$ is a complete Boolean algebra. Then every epimorphism $\pi : F \to G$ in* $\mathbf{Shv}(\mathscr{B})$ *has a section.*

Lemma 2.30 asserts that the sheaf category on a complete Boolean algebra satisfies the Axiom of Choice.

Proof. Consider the family of lifts

$$\begin{tikzcd} & F \arrow[d, "\pi"] \\ N \arrow[ur] \arrow[r, "\leq"'] & G \end{tikzcd}$$

This family is non-empty, because every $x \in G(1)$ restricts along some covering $B \leq 1$ to a family of elements x_B which lift to $F(B)$.

All maps $\hom(\ ,B) \to G$ are monomorphisms, so that all such morphisms represent objects of $Sub(G)$, which is a complete Boolean algebra by Lemma 2.29.

Zorn's Lemma implies that the family of lifts has maximal elements. Suppose that N is maximal and that $\neg N \neq \emptyset$. Then there is an $x \in \neg N(C)$ for some C, and there is a covering $B' \leq C$ such that $x_{B'} \in N(B')$ lifts to $F(B')$ for all members of the cover. Then $N \wedge \hom(\ ,B') = \emptyset$ so the lift extends to a lift on $N \vee \hom(\ ,B')$, contradicting the maximality of N. □

A *Boolean localization* for $\mathbf{Shv}(\mathscr{C})$ is a geometric morphism $p : \mathbf{Shv}(\mathscr{B}) \to \mathbf{Shv}(\mathscr{C})$ such that $\mathscr{B}$ is a complete Boolean algebra and p^* is faithful.

Theorem 2.31 (Barr). *Boolean localizations exist for every Grothendieck topos* $\mathbf{Shv}(\mathscr{C})$.

Theorem 2.31 is one of the big results of topos theory, and is proved in multiple places — see [52], for example. There is a relatively simple proof of this result in the next section.

A Grothendieck topos $\mathbf{Shv}(\mathscr{C})$ may not have enough points, in general (eg. sheaves on the flat site for a scheme), but the result asserts that every Grothendieck topos has a "fat point" given by a Boolean localization.

2.5 Boolean localization

This section contains a relatively short proof of the Barr theorem (Theorem 2.31) which says that every Grothendieck topos has a Boolean cover.

The proof is in two steps, just as in the literature (eg. [52]):

1) Show that every Grothendieck topos has a localic cover.
2) Show that every localic topos has a Boolean cover.

We begin with the second step: the precise statement is Theorem 2.39 below. The first statement is Diaconescu's Theorem, which appears here as Theorem 2.44.

Recall that a frame F is a lattice which has all small joins and satisfies an infinite distributive law. Recall also that every frame A has a canonical Grothendieck topology: say that a family $y_i \leq x$ is covering if $\bigvee_i y_i = x$. Write $\mathbf{Shv}(A)$ for the corresponding sheaf category.

Say that a Grothendieck topos B is *localic* if it is equivalent to $\mathbf{Shv}(A)$ for some frame A.

Theorem 2.32. *A Grothendieck topos $\mathscr{E}$ is localic if and only if it is equivalent to* $\mathbf{Shv}(P)$ *for some topology on a poset P.*

Proof (Outline). The corresponding frame is the poset of subobjects of the terminal object $1 = *$. These subobjects generate $\mathscr{E}$, and then Giraud's Theorem is used to finish the proof. $\square$

A more detailed proof of Theorem 2.32 can be found in [52, IX.5].

A *morphism of frames* is a poset morphism $f : A \to B$ which preserves structure, ie. preserves all finite meets and all infinite joins, hence preserves both 0 and 1.

Lemma 2.33. *Every frame morphism $f : A \to B$ has a right adjoint $f_* : B \to A$.*

Proof. Set $f_*(y) = \bigvee_{f(x)\leq y} x$. $\square$

Suppose that $i : P \to B$ is a morphism of frames. Then precomposition with i determines a functor $i_* : \mathbf{Shv}(B) \to \mathbf{Shv}(P)$, since i preserves covers. The left adjoint

$$i^* : \mathbf{Shv}(P) \to \mathbf{Shv}(B)$$

of i_* associates to a sheaf F the sheaf i^*F, which is the sheaf associated to the presheaf i^pF, where

$$i^pF(x) = \varinjlim_{x\to i(y)} F(y).$$

This colimit is filtered since i preserves meets.

Lemma 2.34. *Suppose that $i : P \to B$ is a morphism of frames and that F is a sheaf on P. Then the presheaf i^pF is separated.*

Proof. Suppose that $\alpha, \beta \in i^pF(x)$ map to the same element in $i^*F(x)$. Then there is a covering family $z_j \leq x$ such that α, β restrict to the same element of $i^pF(z_j)$ for all j.

Identify α and β with representatives $\alpha, \beta \in F(y)$ for some fixed $x \leq i(y)$. For each j there is a commutative diagram of relations

$$\begin{tikzcd}
z_j \arrow[r] \arrow[d] & i(v_j) \arrow[d] \\
x \arrow[r] & i(y)
\end{tikzcd}$$

such that α and β restrict to the same element of $F(v_j)$. But then α and β restrict to the same element of $F(\vee v_j)$ and $\vee z_j = x$ and there is a commutative diagram

$$\begin{tikzcd}
x \arrow[r] \arrow[dr] & i(\vee v_j) \arrow[d] \\
& i(y)
\end{tikzcd}$$

Then F is a sheaf, so that α and β map to the same element of $F(\vee v_j)$x and therefore represent the same element of $i^p F(x)$. □

Lemma 2.35. *Suppose that the frame morphism $i : P \to B$ is a monomorphism. Then the functor $i^* : \mathbf{Shv}(P) \to \mathbf{Shv}(B)$ is faithful.*

Proof. By Lemma 2.34, it is enough to show that the canonical map $\eta : F \to i_* i^p F$ is a monomorphism of presheaves for all sheaves F on P. For then $\eta : F \to i_* i^* F$ is a monomorphism, and so i^* is faithful (exercise).

The map

$$\eta : F(y) \to \varinjlim_{i(y) \leq i(z)} F(z)$$

is the canonical map into the colimit which is associated to the identity map $i(y) \leq i(y)$ of B.

The frame morphism i is a monomorphism, so that $x = i_* i(x)$ for all $x \in P$, where i_* is the right adjoint of $i : P \to B$. Thus, $i(y) \leq i(z)$ if and only if $y \leq z$, so that category of all morphisms $i(y) \leq i(z)$ has an initial object, namely the identity on $i(y)$. The map η is therefore an isomorphism for all y. □

Suppose that P is a frame and $x \in P$. Write P_x for the subobject of P consisting of all y such that $x \leq y$. Then P_x is a frame with initial object x and terminal object 1. There is a frame morphism

$$\phi_x : P \to P_x$$

defined by $\phi_x(w) = x \vee w$.

Suppose that Q is a frame and that $x \in Q$. Write

$$\neg x = \bigvee_{x \wedge y = 0} y$$

Note that $x \wedge \neg x = 0$ so that there is a relation (morphism)

$$\eta : x \leq \neg\neg x$$

for all $x \in Q$; this relation is natural in x. Further, the relation η induces the relation $\neg\eta : \neg\neg\neg x \leq \neg x$, while we have the relation $\eta : \neg x \leq \neg\neg\neg x$ for $\neg x$. It follows that the relation

$$\eta : \neg x \leq \neg\neg\neg x$$

is an equality (isomorphism) for all $x \in Q$.

Define a subposet $\neg\neg Q$ of Q by

$$\neg\neg Q = \{y \in Q |\ y = \neg\neg y\}$$

There is a diagram of relations

$$\begin{tikzcd}
x \wedge y \arrow[r] \arrow[d] & \neg\neg(x \wedge y) \arrow[dl] \\
(\neg\neg x) \wedge (\neg\neg y) &
\end{tikzcd}$$

Thus, the element $x \wedge y$ is a member of $\neg\neg Q$ if both x and y are in $\neg\neg Q$, for in that case the vertical map in the diagram is an isomorphism. If the set of objects x_i are members of $\neg\neg Q$, then the element $\neg\neg(\vee_i x_i)$ is their join in $\neg\neg Q$. It follows that the poset $\neg\neg Q$ is a frame, and that the assigment $x \mapsto \neg\neg x$ defines a frame morphism

$$\gamma : Q \to \neg\neg Q.$$

Lemma 2.36. *The frame $\neg\neg Q$ is a complete Boolean algebra, for every frame Q.*

Proof. Observe that $y \leq \neg z$ if and only if $y \wedge z = 0$. It follows that $\neg(\vee(\neg y_i))$ is the meet $\wedge y_i$ in $\neg\neg Q$, giving the completeness. Also, x is complemented by $\neg x$ in $\neg\neg Q$ since

$$x \vee (\neg x) = \neg\neg x \vee \neg\neg\neg x = \neg(\neg x \wedge x) = \neg 0 = 1.$$

□

Write ω for the composite frame morphism

$$P \xrightarrow{(\phi_x)} \prod_{x \in P} P_x \xrightarrow{(\gamma)} \prod_{x \in P} \neg\neg P_x,$$

and observe that the product $\prod_x \neg\neg P_x$ is a complete Boolean algebra.

Lemma 2.37. *The frame morphism ω is a monomorphism.*

Proof. If $x \leq y$ then $\neg\neg\phi_x(y) = 0$ in P_x implies that there is a relation

$$x \vee y \leq \neg\neg(x \vee y) = x,$$

so that $x \vee y = x$ in P_x and hence $y = x$ in P. Thus, if $x \leq y$ and $y \neq x$ then x and y have distinct images $\omega(x) < \omega(y)$ in $\prod_x \neg\neg P_x$.

Suppose that y and z are distinct elements of P. Then $y \neq y \vee z$ or $z \leq y$ and $z \neq y$. Then $\omega(y) \neq \omega(y) \vee \omega(z)$ or $\omega(z) \leq \omega(y)$ and $\omega(z) \neq \omega(y)$. The assumption that $\omega(y) = \omega(z)$ contradicts both possibilities, so $\omega(y) \neq \omega(z)$. □

Corollary 2.38. *Every frame P admits an imbedding $i : P \to B$ into a complete Boolean algebra.*

We have proved the following:

Theorem 2.39. *Suppose that P is a frame. Then there is a complete Boolean algebra B, and a topos morphism $i : \mathbf{Shv}(B) \to \mathbf{Shv}(P)$ such that the inverse image functor $i^* : \mathbf{Shv}(P) \to \mathbf{Shv}(B)$ is faithful.*

A geometric morphism i as in the statement of the Theorem is called a *Boolean cover* of $\mathbf{Shv}(P)$.

Suppose that $\mathscr{C}$ is a (small) Grothendieck site. Write $\mathbf{St}(\mathscr{C})$ for the poset of all finite strings

$$\sigma : x_n \to \cdots \to x_0$$

where $\tau \leq \sigma$ if τ extends σ to the left in the sense that τ is of the form

$$y_k \to \cdots \to y_{m+1} \to x_n \to \cdots \to x_0$$

There is a functor $\pi : \mathbf{St}(\mathscr{C}) \to \mathscr{C}$ which is defined by $\pi(\sigma) = x_n$ for σ as above.

If $R \subset \hom(\ ,\sigma)$ is a sieve of $\mathbf{St}(\mathscr{C})$, then $\pi(R) \subset \hom(\ ,x_n)$ is a sieve of $\mathscr{C}$. In effect, if $\tau \leq \sigma$ is in R and $z \to y_k$ is morphism of $\mathscr{C}$, then the string

$$\tau_* : z \to y_k \to \cdots \to y_{m+1} \to x_n \to \dots x_0$$

refines τ and the relation $\tau_* \leq \tau$ maps to $z \to y_k$.

Say that a sieve $R \subset \hom(\ ,\sigma)$ is covering if $\pi(R)$ is a covering sieve of $\mathscr{C}$. Then $\mathbf{St}(\mathscr{C})$ acquires the structure of a Grothendieck site.

Lemma 2.40. *Suppose that F is a sheaf of sets on $\mathscr{C}$. Then $\pi^*(F) = F \cdot \pi$ is a sheaf on $\mathbf{St}(\mathscr{C})$.*

The proof of this result is an exercise.

Lemma 2.41. *The functor $F \mapsto \pi^*(F)$ is faithful.*

Proof. For $x \in \mathscr{C}$ let $\{x\}$ denote the corresponding string of length 0. Then we have $\pi^*F(\{x\}) = F(x)$. If sheaf morphisms $f,g : F \to G$ on $\mathscr{C}$ induce maps $f_*, g_* : \pi^*(F) \to \pi^*(G)$ such that $f_* = g_*$, then $f_* = g_* : \pi^*F(\{x\}) \to \pi^*G(\{x\})$ for all $x \in \mathscr{C}$. This means, then, that $f = g$. □

Lemma 2.42. *The functor π^* preserves local epimorphisms and local monomorphisms of presheaves.*

Proof. Suppose $m : P \to Q$ is a local monomorphism of presheaves on $\mathscr{C}$. This means that if $m(\alpha) = m(\beta)$ for $\alpha, \beta \in P(x)$ there is a covering $\phi_i : y_i \to x$ such that $\phi_i^*(\alpha) = \phi_i^*(\beta)$ for all ϕ_i.

Suppose that $\alpha, \beta \in \pi^*P(\sigma)$ such that $m_*(\alpha) = m_*(\beta)$ in $\pi^*Q(\sigma)$. Then $\alpha, \beta \in P(x_n)$ and $m(\alpha) = m(\beta) \in Q(x_n)$. There is a covering $\phi_i : y_i \to x_n$ such that $\phi_i^*(\alpha) = \phi_i^*(\beta)$ for all ϕ_i. But then α, β map to the same element of

$$\pi^*P(y_i \to x_n \to \cdots \to x_0)$$

for all members of a cover of σ.

Suppose that $p : P \to Q$ is a local epimorphism of presheaves on $\mathscr{C}$. Then for all $\alpha \in Q(x)$ there is a covering $\phi_i : y_i \to x$ such that $\phi_i^*(\alpha)$ lifts to an element of $P(y_i)$

for all i. Given $\alpha \in \pi^* Q(\sigma)$, $\alpha \in Q(x_n)$, and there is a cover $\phi_i : y_i \to x_n$ such that $\phi_i^*(\alpha)$ lifts to $P(y_i)$. It follows that there is a cover of σ such that the image of α in

$$\pi^* Q(y_i \to x_n \to \cdots \to x_0)$$

lifts to

$$\pi^* P(y_i \to x_n \to \cdots \to x_0)$$

for all members of the cover. □

Lemma 2.43. *The functor*

$$\pi^* : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Shv}(\mathbf{St}(\mathscr{C}))$$

preserves all small colimits.

Proof. Suppose that $A : I \to \mathbf{Shv}(\mathscr{C})$ is a small diagram of sheaves. Write $\varinjlim_i A$ for the presheaf colimit, and let

$$\eta : \varinjlim A \to L^2(\varinjlim A)$$

be the natural associated sheaf map. The map η is a local epimorphism and a local monomorphism. The functor π^* plainly preserves presheaf colimits, and there is a diagram

$$\begin{tikzcd}
\pi^*(\varinjlim A) \arrow[r, "\pi^*(\eta)"] & \pi^*(L^2(\varinjlim A)) \\
\varinjlim \pi^* A \arrow[u, "\cong"] \arrow[r, "\eta"'] & L^2(\varinjlim \pi^* A) \arrow[u]
\end{tikzcd}$$

Then $\pi^*(\eta)$ is a local epimorphism and a local monomorphism by Lemma 2.42. It follows that the map

$$L^2(\varinjlim \pi^* A) \to \pi^*(L^2(\varinjlim A))$$

is a local epimorphism and a local monomorphism of sheaves (use Lemma 2.40), and is therefore an isomorphism. □

It is a consequence of the following result that any Grothendieck topos has a localic cover (see also [52, IX.9]). The topos $\mathbf{Shv}(\mathbf{St}(\mathscr{C}))$ is also called the *Diaconescu cover* of $\mathbf{Shv}(\mathscr{C})$.

Theorem 2.44 (Diaconescu). *The right adjoint $\pi_* : \mathbf{Pre}(\mathbf{St}(\mathscr{C})) \to \mathbf{Pre}(\mathscr{C})$ of precomposition with π restricts to a functor*

$$\pi_* : \mathbf{Shv}(\mathbf{St}(\mathscr{C})) \to \mathbf{Shv}(\mathscr{C})$$

which is right adjoint to π^*. The functors π^* and π_* determine a geometric morphism

$$\pi : \mathbf{Shv}(\mathbf{St}(\mathscr{C})) \to \mathbf{Shv}(\mathscr{C}).$$

The functor π^* is faithful.

Proof. A covering sieve $R \subset \hom(\ ,x)$ in $\mathscr{C}$ determines an isomorphism of sheaves

$$\varinjlim_{y \to x} L^2 \hom(\ ,y) \cong L^2 \hom(\ ,x).$$

The functor π^* preserves colimits of sheaves, and so $\pi_* G$ is a sheaf if G is a sheaf. The functor π^* plainly preserves finite limits, so that the functors π^* and π_* form a geometric morphism. The last statement is Lemma 2.41. □

We have thus assembled a proof of Barr's Theorem (Theorem 2.31), which can be restated as follows:

Theorem 2.45. *Suppose that $\mathscr{C}$ is a small Grothendieck site. Then there are geometric morphisms*

$$\mathbf{Shv}(B) \xrightarrow{f} \mathbf{Shv}(\mathbf{St}(\mathscr{C})) \xrightarrow{\pi} \mathbf{Shv}(\mathscr{C})$$

such that the inverse image functors f^* and π^* are faithful, and such that B is a complete Boolean algebra.

Chapter 3
Rigidity and the isomorphism conjecture

Suppose that k is an algebraically closed field and let ℓ be a prime which is distinct from the characteristic of k.

We will be working with the big étale site $(Sch|_k)_{et}$ over the field k throughout this section. Note the abuse: one should write $(Sch|_{\mathrm{Sp}(k)})_{et}$ for this object.

We shall use the notation Gl_n to represent either the algebraic group

$$Gl_n = \mathrm{Sp}(k[X_{ij}]_{det})$$

over k, or the sheaf of groups

$$Gl_n = \hom(\ ,Gl_n)$$

that it represents on the big site $(Sch|_k)_{et}$.

Observe that Gl_1 is the multiplicative group $\mathbb{G}_m$. One often sees the notation $\mu = \mathbb{G}_m$, and μ_ℓ for its ℓ-torsion part. Since the prime ℓ is distinct from the characteristic of the algebraically closed field k, there is an isomorphism of sheaves

$$\mu_\ell \cong \Gamma^*\mathbb{Z}/\ell = \mathbb{Z}/\ell,$$

where $\Gamma^*\mathbb{Z}/\ell$ is the constant sheaf on the cyclic group $\mathbb{Z}/\ell$ and the displayed equality is again a standard abuse.

In general, the constant sheaf functor $A \mapsto \Gamma^*(A)$ is left adjoint to the global sections functor $X \mapsto \Gamma_* X$, where

$$\Gamma_* X = X(k),$$

and there's a geometric morphism

$$\Gamma : \mathbf{Shv}((Sch|_k)_{et}) \to \mathbf{Set}.$$

This is a special case of a geometric morphism

$$\Gamma : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Set}$$

which is defined for an arbitrary site $\mathscr{C}$, where

$$\Gamma_* X = \varprojlim_{U \in \mathscr{C}} X(U),$$

defines the global sections functor for an arbitrary site $\mathscr{C}$. This general version of Γ_* specializes to the functor defined above for sheaves on $(Sch|_k)_{et}$ because this site has a terminal object, namely $\mathrm{Sp}(k)$.

Remark 3.1. It is a special feature of étale sites (and some others) that

$$\Gamma^* A(U) = \hom(\pi_0 U, A)$$

where $\pi_0(U)$ is the set of connected components of the k-scheme U, since $\mathrm{Sp}(k)$ is connected. In effect, the k-scheme $\bigsqcup_A \mathrm{Sp}(k)$ represents $\Gamma^* A$, and there is an easily proved isomorphism

$$\hom_k(U, \bigsqcup_A \mathrm{Sp}(k)) \cong \hom(\pi_0 U, A).$$

Every k-scheme X represents a sheaf on $(Sch|_k)_{et}$, by the theorem of *faithfully flat descent* — this result can be found in any of the étale cohomology textbooks, such as [12],[53].

In particular, the sheaf of groups Gl_n is defined on affine k-schemes $Sp(R)$ (ie. k-algebras R) by

$$Gl_n(\mathrm{Sp}(R)) = Gl_n(R),$$

where the object on the right is the usual group of invertible $n \times n$ matrices with entries in R. There is a standard way to recover the sheaf Gl_n on $(Sch|_k)_{et}$ from the matrix group description for affine schemes, by an equivalence

$$\mathbf{Shv}((Sch|_k)_{et}) \simeq \mathbf{Shv}((\mathrm{Aff}|_k)_{et})$$

where $(\mathrm{Aff}|_k)_{et}$ is the étale site of affine k-schemes.

The matrix group homomorphisms $Gl_n(R) \to Gl_{n+1}(R)$ defined by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

define a homomorphism $Gl_n \to Gl_{n+1}$ of sheaves of groups. The colimit presheaf

$$Gl = \varinjlim_n Gl_n \tag{3.1}$$

has the traditional infinite general linear group $Gl(R)$ in affine sections.

One typically also writes Gl for the associated sheaf, so that there is a relation of the form (3.1) in the category of sheaves of groups.

Sheaves of groups G have classifying simplicial sheaves BG, with

$$BG(U) = B(G(U))$$

given by the standard simplicial set construction in sections. The object BG is a simplicial sheaf if G is a sheaf, because of the identification

$$BG_n = G \times \cdots \times G$$

(n factors) and the fact that any product of sheaves is a sheaf.

The classifying space construction commutes with filtered colimits, so we are entitled to a classifying simplicial sheaf (or presheaf) BGl with

$$BGl = \varinjlim_n BGl_n.$$

In general, simplicial sheaves (or presheaves) X have cohomology groups and homology sheaves.

The homology sheaves $\tilde{H}_n(X,A)$ are easier to define: form the presheaf of chain complexes

$$\mathbb{Z}(X) \otimes A,$$

with

$$(\mathbb{Z}(X) \otimes A)(U) = \mathbb{Z}(X(U)) \otimes A(U),$$

where $\mathbb{Z}(X(U))$ is the standard Moore chain complex for the simplicial set $X(U)$. Then the *homology sheaf* $\tilde{H}_n(X,A)$ is the sheaf which is associated to the presheaf

$$H_n(\mathbb{Z}(X) \otimes A).$$

Cohomology has a more interesting definition: the *cohomology group* $H^n(X,A)$ of the simplicial presheaf X with coefficients in the abelian presheaf A is defined by

$$H^n(X,A) = [X, K(A,n)],$$

where the object on the right is morphisms in the local homotopy category of simplicial presheaves on the étale site.

There is a model structure on simplicial presheaves (respectively, and Quillen equivalently, simplicial sheaves) on the site $(Sch|_k)_{et}$, for which the weak equivalences are those maps $X \to Y$ which induce weak equivalences of simplicial sets in all stalks — I call these *local weak equivalences* , and for which the cofibrations are the monomorphisms. This is a special case of a construction which holds for arbitrary Grothendieck sites.

Example 3.2. The canonical map $\eta : X \to \tilde{X}$ from a simplicial presheaf to its associated simplicial sheaf is a local weak equivalence.

The simplicial presheaf $K(A,n)$ is the diagonal of the n-fold simplicial presheaf $B^n(A)$, which is constructed by n iterated applications of the classifying space functor B to all constitutent abelian groups. Equivalently, $K(A,n)$ is the presheaf $\Gamma(A[-n])$, where Γ is the Dold-Kan functor from chain complexes to simplicial abelian groups, and $A[-n]$ is the presheaf of chain complexes which consists of a copy of A concentrated in degree n.

If X is represented by a (simplicial) scheme having the same name, and A is a sheaf of abelian groups, then $H^n(X,A)$ coincides up to isomorphism with the étale cohomology group $H^n_{et}(X,A)$ of X, as it is normally defined.

In particular, if X is a k-scheme, and $A \to I^*$ is an injective resolution of A in sheaves of abelian groups, then there is an isomorphism

$$H^n(X,A) \cong H^n(I^*(X)) \cong \operatorname{Ext}^n(\mathbb{Z}(X),A).$$

The homotopy theoretic description of cohomology which is displayed above generalizes the standard definition of étale cohomology groups of schemes to arbitrary simplicial presheaves.

There is a spectral sequence relating homology sheaves and cohomology groups, with

$$E_2^{p,q} = \operatorname{Ext}^p(\tilde{H}_q(X),A) \Rightarrow H^{p+q}(X,A).$$

There is also an ℓ-torsion version, with

$$E_2^{p,q} = \operatorname{Ext}^p(\tilde{H}_q(X,\mathbb{Z}/\ell),A) \Rightarrow H^{p+q}(X,A) \tag{3.2}$$

if A is an ℓ-torsion sheaf.

It follows that if $f : X \to Y$ is a map of simplicial presheaves which induces homology sheaf isomorphisms

$$f_* : \tilde{H}_n(X,\mathbb{Z}/\ell) \xrightarrow{\cong} \tilde{H}_n(Y,\mathbb{Z}/\ell),\ n \geq 0,$$

then f induces isomorphisms

$$f^* : H^n(Y,\mathbb{Z}/\ell) \xrightarrow{\cong} H^n(X,\mathbb{Z}/\ell)$$

in étale cohomology groups for all $n \geq 0$.

Exercise 3.3. Show that if $p : F \to F'$ is a local epimorphism of presheaves on $(Sch|_k)_{et}$, then the induced map $F(k) \to F'(k)$ in global sections is surjective, since k is an algebraically closed field.

It follows that the associated sheaf map $\eta : F \to \tilde{F}$ induces a bijection $F(k) \xrightarrow{\cong} \tilde{F}(k)$ in global sections.

It also follows that the global sections functor on $\mathbf{Shv}((Sch|_k)_{et})$ is exact on abelian sheaves. In particular, there are isomorphisms

$$H^n_{et}(k,A) \cong \begin{cases} A(k) & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

More generally, the map $A \to I^*$ of chain complexes defined by an injective resolution with I^* is in negative degrees induces a natural isomorphism

$$H^n(X,A(k)) \cong H^n(\Gamma^* X,A) \tag{3.3}$$

for any simplicial set X and sheaf of abelian groups A.

The canonical map

$$\varepsilon : \Gamma^*\Gamma_* BGl \to BGl$$

has the form

$$\varepsilon : \Gamma^* BGl(k) \to BGl$$

up to isomorphism, and the identification (3.3) implies that the induced map

$$\varepsilon^* : H^n(BGl,\mathbb{Z}/\ell) \to H^n(\Gamma^* BGl(k),\mathbb{Z}/\ell)$$

can be written as

$$\varepsilon^* : H^n_{et}(BGl,\mathbb{Z}/\ell) \to H^n(BGl(k),\mathbb{Z}/\ell), \tag{3.4}$$

where the object on the right is a standard cohomology group of the simplicial set $BGl(k)$ with coefficients in the abelian group $\mathbb{Z}/\ell$.

The map (3.4) is a comparison map of étale with discrete cohomology for the general linear group Gl.

Much of local homotopy theory evolved from the enabling technology for the proof of the following result:

Theorem 3.4. *Suppose that k is an algebraically closed field, and that ℓ is prime which is distinct from the characteristic of k. Then the comparison map*

$$\varepsilon^* : H^n_{et}(BGl,\mathbb{Z}/\ell) \to H^n(BGl(k),\mathbb{Z}/\ell)$$

is an isomorphism.

Remark 3.5. This theorem gives a calculation

$$H^*(BGl(k),\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1,c_2,\dots],$$

since standard results in étale cohomology theory imply that $H^*_{et}(BGl,\mathbb{Z}/\ell)$ is a polynomial ring in Chern classes c_i, with $deg(c_i) = 2i$.

Proof (Proof of Theorem 3.4). The idea is to show that the map ε induces isomorphisms

$$\tilde{H}_n(\Gamma^* BGl(k),\mathbb{Z}/\ell) \xrightarrow{\cong} \tilde{H}_n(BGl,\mathbb{Z}/\ell)$$

in all homology sheaves, and then invoke a comparison of spectral sequences (3.2).

The category $\mathbf{Shv}((Sch|_k)_{et})$ has a good theory of stalks, and it's enough to compare stalks at all closed points $x \in U$ of all k-schemes U (which are locally of finite type over k). The map ε_* at the stalk for such a point x is the map

$$H_n(BGl(k),\mathbb{Z}/\ell) \to H_n(BGl(\mathscr{O}^{sh}_x),\mathbb{Z}/\ell),$$

where $\mathscr{O}_x^{sh}$ is the strict Henselization of the local ring $\mathscr{O}_x$ of U at x, and the indicated map is induced by the k-algebra structure map $k \to \mathscr{O}_x^{sh}$.

The Gabber Rigidity Theorem [19], [21] asserts that the residue field homomorphism $\pi : \mathscr{O}_x^{sh} \to k$ induces an isomorphism

$$\pi_* : H_n(BGl(\mathscr{O}_x^{sh}), \mathbb{Z}/\ell) \xrightarrow{\cong} H_n(BGl(k), \mathbb{Z}/\ell).$$

The Theorem follows.

The Gabber Rigidity Theorem is equivalent to a mod ℓ K-theory rigidity statement, which asserts that the residue map induces isomorphisms

$$\pi_* : K_*(\mathscr{O}_x^{sh}, \mathbb{Z}/\ell) \xrightarrow{\cong} K_*(k, \mathbb{Z}/\ell)$$

As such, it is an essentially stable statement that very much depends on the existence of the K-theory transfer, as well as the homotopy property for algebraic K-theory ($K_*(A) \cong K_*(A[t])$ for regular rings A).

An axiomatic approach to rigidity has evolved in the intervening years, which first appeared in [61], and achieved its modern form for torsion presheaves with transfers satisfying the homotopy property in [62].

Theorem 3.4 implies that an inclusion of algebraically closed fields $k \to L$ of characteristic away from ℓ induces an isomorphism

$$i^* : H^*(BGl(L), \mathbb{Z}/\ell) \cong H^*(BGl(k), \mathbb{Z}/\ell), \tag{3.5}$$

since there is an isomorphism of the corresponding étale cohomology rings by a smooth base change argument. The map i^* is an isomorphism if and only if the map

$$i_* : K_*(k, \mathbb{Z}/\ell) \to K_*(L, \mathbb{Z}/\ell)$$

is an isomorphism, by H-space tricks, so that Theorem 3.4 implies Suslin's first rigidity theorem [60].

The proof of Suslin's second rigidity theorem, for local fields [63], uses Gabber rigidity explicitly. One outcome of that result, that there are isomorphisms

$$K_n(\mathbb{C}, \mathbb{Z}/\ell) \cong \pi_n KU/\ell$$

for $n \geq 0$, is also a consequence of Theorem 3.4.

The comparison map

$$\varepsilon^* : H^n_{et}(BGl, \mathbb{Z}/\ell) \to H^n(BGl(k), \mathbb{Z}/\ell)$$

is a special case of a natural comparison map

$$\varepsilon^* : H^n(X, \mathbb{Z}/\ell) \to H^n(X(k), \mathbb{Z}/\ell)$$

which one can can construct for an arbitrary simplicial presheaf X on the big site $(Sch|_k)_{et}$.

There are versions of Theorem 3.4 for all of the classical infinite families of algebraic groups. In particular, there are comparison isomorphisms

$$\begin{aligned}
\varepsilon^* &: H^*_{et}(BSl,\mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BSl(k),\mathbb{Z}/\ell),\\
\varepsilon^* &: H^*_{et}(BSp,\mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BSp(k),\mathbb{Z}/\ell),\\
\varepsilon^* &: H^*_{et}(BO,\mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BO(k),\mathbb{Z}/\ell),
\end{aligned}$$

for the infinite special linear, symplectic and orthogonal groups, respectively. The special linear case follows from Theorem 3.4, by a fibre sequence argument. The symplectic and orthogonal group statements follow from a rigidity statement for Karoubi L-theory which is deduced from Gabber rigidity with a Karoubi peridicity argument [48].

There is also a comparison map

$$\varepsilon^* : H^n_{et}(BG,\mathbb{Z}/\ell) \to H^n(BG(k),\mathbb{Z}/\ell) \tag{3.6}$$

for an arbitrary algebraic group G over k. Friedlander's generalized isomorphism conjecture asserts that this comparison map is an isomorphism if G is reductive. One says "generalized" because the conjecture specializes to a conjecture of Milnor when the underlying field is the complex numbers, in which case the étale cohomology groups $H^n(BG,\mathbb{Z}/\ell)$ correspond with the ordinary singular cohomology groups of the (simplicial analytic) classifying space $BG(\mathbb{C})$.

The isomorphism conjecture holds when $k=\overline{\mathbb{F}}_p$ is the algebraic closure of the finite field $\mathbb{F}_p$ with $p\neq\ell$ — this is a result of Friedlander and Mislin [17] which depends strongly on the Lang isomorphism for algebraic groups defined over $\mathbb{F}_p$. The isomorphism conjecture is not known to hold, in general, for any other algebraically closed field. It is not even known to hold for any of the general linear groups Gl_n outside of a stable range in homology. See Kevin Knudson's book [50] for a description of the current state of the problem.

This conjecture is perhaps the most important unsolved classical problem of algebraic K-theory. It was known since the 1970s that a calculation of the form

$$H^*(BGl_n(k),\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[c_1,\dots,c_n]$$

would imply the Lichtenbaum conjecture that

$$K_*(k,\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\beta]$$

where $\beta\in K_2(k,\mathbb{Z}/\ell)$ is the Bott element. Suslin proved this conjecture with the stable calculations of [60], [63] which were referred to above, but the unstable problem remains open.

Part II
Simplicial presheaves and simplicial sheaves

These chapters give an introduction to the unstable homotopy theory of simplicial presheaves and sheaves, localized theories and cocycles.

Chapter 4
Local weak equivalences

This chapter describes the first principles of local homotopy theory.

4.1 Local weak equivalences

Suppose that $\mathscr{C}$ is a small Grothendieck site. The notations $s\mathbf{Pre}(\mathscr{C})$ and $s\mathbf{Shv}(\mathscr{C})$ denote the categories of simplicial presheaves and simplicial sheaves on $\mathscr{C}$, respectively.

Recall that a simplicial set map $f : X \to Y$ is a weak equivalence if and only if the induced map $|X| \to |Y|$ is a weak equivalence of topological spaces in the classical sense. This is equivalent to the assertion that all induced morphisms

a) $\pi_0 X \to \pi_0 Y$, and
b) $\pi_n(X,x) \to \pi_n(Y, f(x))$, $x \in X_0, n \geq 1$

are bijections.

One could define $\pi_n(X,x) = \pi_n(|X|,x)$ in general. We also have an identification

$$\pi_n(X,x) = [(S^n,*),(X,x)]$$

with pointed homotopy classes of maps, where $S^n = \Delta^n/\partial\Delta^n$ is the simplicial n-sphere. Finally, if X is a Kan complex, then we have

$$\pi_n(X,x) = \pi((S^n,*),(X,x)) \tag{4.1}$$

by the Milnor theorem (Theorem 1.19), where $\pi((S^n,*),(X,x))$ is pointed simplicial homotopy classes of maps.

Write

$$\pi_n X = \bigsqcup_{x \in X_0} \pi_n(X,x)$$

for a Kan complex X. Then the canonical function $\pi_n X \to X_0$ gives $\pi_n X$ the structure of a group object over X_0, which is abelian if $n \geq 2$.

Then one verifies easily that a map $f : X \to Y$ of simplicial sets is a weak equivalence if the following hold:

a) $\pi_0 X \to \pi_0 Y$ is a bijection, and
b) all diagrams

$$\begin{tikzcd} \pi_n X \arrow[r] \arrow[d] & \pi_n Y \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

are pullbacks for $n \geq 1$.

If X is a Kan complex, then the object $\pi_n(X,x)$ can also be defined as a set by setting

$$\pi_n(X,x) = \pi_0 F_n(X)_x,$$

where $F_n(X)_x$ is defined by the pullback diagram

$$\begin{tikzcd} F_n(X)_x \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^n, X) \arrow[d, "i^*"] \\ \Delta^0 \arrow[r, "x"'] & \mathbf{hom}(\partial\Delta^n, X) \end{tikzcd}$$

in which i^* is the Kan fibration between function spaces which is induced by the inclusion $i : \partial\Delta^n \subset \Delta^n$. Define the space $F_n(X)$ by the pullback diagram

$$\begin{tikzcd} F_n(X) \arrow[r] \arrow[d] & \mathbf{hom}(\Delta^n, X) \arrow[d, "i^*"] \\ X_0 \arrow[r, "\alpha"'] & \mathbf{hom}(\partial\Delta^n, X) \end{tikzcd} \tag{4.2}$$

where $\alpha(x)$ is the constant map $\partial\Delta^n \to \Delta^0 \xrightarrow{x} X$ at the vertex x. Then

$$F_n(X) = \bigsqcup_{x \in X_0} F_n(X)_x,$$

so that

$$\pi_0 F_n(X) = \bigsqcup_{x \in X_0} \pi_0 F_n(X)_x = \bigsqcup_{x \in X_0} \pi_n(X,x) = \pi_n X.$$

as object fibred over X_0.

It follows that if X and Y are Kan complexes, then $f : X \to Y$ is a weak equivalence if and only if

a) the induced function $\pi_0 X \to \pi_0 Y$ is a bijection, and
b) all diagrams

$$\begin{tikzcd} \pi_0 F_n(X) \arrow[r] \arrow[d] & \pi_0 F_n(Y) \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

are pullbacks for $n \geq 1$.

Kan's Ex^∞ construction gives a natural combinatorial method of replacing a simplicial set by a Kan complex up to weak equivalence.

The functor $\mathrm{Ex} : s\mathbf{Set} \to s\mathbf{Set}$ is defined by

$$\mathrm{Ex}(X)_n = \hom(\mathrm{sd}\,\Delta^n, X).$$

$\mathrm{sd}\,\Delta^n = BN\Delta^n$, where $N\Delta^n$ is the poset of non-degenerate simplices of Δ^n (subsets of $\{0,1,\dots,n\}$). Any ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ induces a functor $N\Delta^m \to N\Delta^n$, and hence induces a simplicial set map $\mathrm{sd}\,\Delta^m \to \mathrm{sd}\,\Delta^n$. Precomposition with this map gives the simplicial structure map θ^* of $\mathrm{Ex}(X)$. There is a last vertex functor $N\Delta^n \to \mathbf{n}$, which is natural in ordinal numbers $\mathbf{n}$; the collection of such functors determines a natural simplicial set map

$$\eta : X \to \mathrm{Ex}(X).$$

Observe that $\mathrm{Ex}(X)_0 = X_0$, and that η induces a bijection on vertices.

Iterating gives a simplicial set $\mathrm{Ex}^\infty(X)$ which is defined by the assignment

$$\mathrm{Ex}^\infty(X) = \varinjlim \mathrm{Ex}^n(X),$$

and a natural map $j : X \to \mathrm{Ex}^\infty(X)$.

The salient features of the construction are the following (see [24, III.4]):

1) the map $\eta : X \to \mathrm{Ex}(X)$ is a weak equivalence,
2) the functor $X \mapsto \mathrm{Ex}(X)$ preserves Kan fibrations
3) the simplicial set $\mathrm{Ex}^\infty(X)$ is a Kan complex, and the natural map $j : X \to \mathrm{Ex}^\infty(X)$ is a weak equivalence.

It follows that a simplicial set map $f : X \to Y$ is a weak equivalence if and only if the induced map $f_* : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(Y)$ is a weak equivalence, so that f is a weak equivalence if and only if

a) the function $\pi_0 X \to \pi_0(Y)$ is a bijection, and
b) the diagram

$$\begin{tikzcd} \pi_0 F_n(\mathrm{Ex}^\infty(X)) \arrow[r] \arrow[d] & \pi_0 F_n(\mathrm{Ex}^\infty(Y)) \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

is a pullback for $n \geq 1$.

Observe that $X_0 = \mathrm{Ex}^\infty X_0$, and the set $\pi_0 F_n(\mathrm{Ex}^\infty(X))$ is a disjoint union of simplicial homotopy groups $\pi_n(\mathrm{Ex}^\infty X, x)$ for $x \in X_0$.

We shall define the n^{th} homotopy group object $\pi_n X$ of a simplicial set X by setting

$$\pi_n X = \pi_0 F_n(\mathrm{Ex}^\infty(X))$$

in all that follows.

The fundamental idea of local homotopy theory is that the topology of the underlying site $\mathscr{C}$ should create weak equivalences.

It is relatively easy to see what the local weak equivalences should look like for simplicial presheaves on a topological space: a map $f : X \to Y$ of simplicial presheaves on $op|_T$ for a topological space T should be a local weak equivalence if and only if it induces a weak equivalence of simplicial sets $X_x \to Y_x$ in stalks for all $x \in T$. In particular f should induce isomorphisms

$$\pi_n(X_x, y) \to \pi_n(Y_x, f(y))$$

for all $n \geq 1$ *and all choices of base point* $y \in X_x$ and $x \in T$, as well as bijections

$$\pi_0 X_x \xrightarrow{\cong} \pi_0 Y_x.$$

Recall that the stalk

$$X_x = \varinjlim_{x \in U} X(U)$$

is a filtered colimit, and so each base point y of X_x comes from somewhere, namely some $z \in X(U)$ for some U. The point z determines a global section of $X|_U$, where the restriction $X|_U$ is the composite

$$((op|_T)/U)^{op} \to (op|_T)^{op} \xrightarrow{X} s\mathbf{Set}.$$

The map f restricts to a simplicial presheaf map $f|_U : X|_U \to Y|_U$. Then one can show that f is a weak equivalence in all stalks if and only if all induced maps

a) $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$, and
b) $\tilde{\pi}_n(X|_U, z) \to \tilde{\pi}_n(Y|_U, f(z))$, for all $n \geq 1$, $U \in \mathscr{C}$, and $z \in X_0(U)$

in associated sheaves.

This is equivalent to the following: the map $f : X \to Y$ of simplicial presheaves on the topological space T is a local weak equivalence if and only if

a) $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism
b) the presheaf diagrams

$$\begin{tikzcd} \pi_n X \arrow[r] \arrow[d] & \pi_n Y \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

induce pullback diagrams of associated sheaves for $n \geq 1$.

These last two descriptions generalize to equivalent sets of conditions for maps of simplicial presheaves on an arbitrary site $\mathscr{C}$, but the equivalence requires proof. We begin with the following:

Definition 4.1. A map $f : X \to Y$ of $s\mathbf{Pre}(\mathscr{C})$ is a *local weak equivalence* if and only if

a) the map $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism of sheaves, and
b) the diagrams

$$\begin{tikzcd} \pi_n X \arrow[r] \arrow[d] & \pi_n Y \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

induce pullback diagrams of associated sheaves for $n \geq 1$.

Suppose that X_0 is a presheaf and that $x : * \to X_0$ is a global section of X_0. Suppose that $X \to X_0$ is a presheaf morphism, and define a presheaf $X(x)$ by the pullback diagram

$$\begin{tikzcd} X(x) \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r, "x"'] & X_0 \end{tikzcd}$$

The *restriction* $X|_U$ of a presheaf X to the site $\mathscr{C}/U$ is the composite

$$(\mathscr{C}/U)^{op} \to \mathscr{C}^{op} \xrightarrow{X} s\mathbf{Set}.$$

Lemma 4.2. *Suppose given a commutative diagram of presheaves*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & W \arrow[d] \\ Z_0 \arrow[r, "f"'] & W_0 \end{tikzcd} \tag{4.3}$$

Then the induced diagram of associated sheaves is a pullback if and only if the maps

$$L^2(Z|_U)(x)) \to L^2(W|_U)(f(x))$$

are isomorphisms of sheaves for all $x \in Z_0(U)$ and $U \in \mathscr{C}$.

Corollary 4.3. *A map $f : X \to Y$ of simplicial presheaves on $\mathscr{C}$ is a local weak equivalence if and only if*

1) the map $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ is an isomorphism of sheaves, and
2) all induced maps $\tilde{\pi}_n(X|_U, x) \to \tilde{\pi}_n(Y|_U, f(x))$ are isomorphisms of sheaves on $\mathscr{C}/U$ for all $n \geq 1$, all $U \in \mathscr{C}$, and all $x \in X_0(U)$.

Proof (Proof of Lemma 4.2). Suppose that $x \in Z_0(U)$ is a section of Z_0 and form the pullback diagram

$$\begin{tikzcd} Z_{U,x} \arrow[r] \arrow[d] & Z \arrow[d] \\ U \arrow[r, "x"'] & Z_0 \end{tikzcd}$$

in presheaves, where $U = \hom(\ ,U)$ is the presheaf represented by $U \in \mathscr{C}$. Then there is an isomorphism

$$\varinjlim_{U \xrightarrow{x} Z_0} Z_{U,x} \xrightarrow{\cong} Z$$

which is natural in presheaves over Z_0, and hence an isomorphism

$$\varinjlim_{U \xrightarrow{x} Z_0} \tilde{Z}_{U,x} \xrightarrow{\cong} \tilde{Z}$$

in sheaves over $\tilde{Z}_0$. The diagrams

$$\begin{tikzcd} \tilde{Z}_{U,x} \arrow[r] \arrow[d] & \tilde{Z} \arrow[d] \\ \tilde{U} \arrow[r, "x"'] & \tilde{Z}_0 \end{tikzcd}$$

of associated sheaves are also pullbacks.

It follows that the diagram (4.3) induces a pullback diagram of sheaves if and only if all sheaf maps

$$\tilde{Z}_{U,x} \to \tilde{W}_{U,f(x)}$$

are isomorphisms.

Write r^* for the left adjoint of the restriction functor $X \mapsto X|_U$, for both presheaves and sheaves. There is a natural isomorphism of presheaves

$$r^*(X|_U(x)) \cong X_{U,x},$$

and hence a natural isomorphism of sheaves

$$r^*L^2(X|_U(x)) \cong \tilde{X}_{U,x}. \tag{4.4}$$

Restriction commutes with formation of the associated sheaf, and preserves pullbacks. Thus, if the diagram of sheaves

$$\begin{tikzcd} \tilde{Z} \arrow[r] \arrow[d] & \tilde{W} \arrow[d] \\ \tilde{Z}_0 \arrow[r, "f"'] & \tilde{W}_0 \end{tikzcd}$$

is a pullback, then the diagram

$$\begin{tikzcd} L^2(Z|_U) \arrow[r] \arrow[d] & L^2(W|_U) \arrow[d] \\ L^2(Z_0|_U) \arrow[r, "f"'] & L^2(W_0|_U) \end{tikzcd}$$

is a pullback, and it follows that the map

$$L^2(Z|_U)(x)) \to L^2(W|_U)(f(x)) \tag{4.5}$$

is an isomorphism of sheaves.

If the map (4.5) is an isomorphism, then the map

$$\tilde{Z}_{U,x} \to \tilde{W}_{U,f(x)}$$

is an isomorphism on account of the identification (4.4).

The following result gives a first example:

Lemma 4.4. *Suppose that $f : X \to Y$ is a sectionwise weak equivalence in the sense that all $X(U) \to Y(U)$ are weak equivalences of simplicial sets. Then f is a local weak equivalence.*

Proof. The map $\pi_0 X \to \pi_0 Y$ is an isomorphism of presheaves, and all diagrams

$$\begin{tikzcd} \pi_n X \arrow[r] \arrow[d] & \pi_n Y \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

are pullbacks of presheaves. Apply the associated sheaf functor.

The Ex^∞ construction extends to a construction for simplicial presheaves, which construction preserves and reflects local weak equivalences:

Lemma 4.5. *A map $f : X \to Y$ of simplicial presheaves is a local weak equivalence if and only if the induced map $\mathrm{Ex}^\infty X \to \mathrm{Ex}^\infty Y$ is a local weak equivalence.*

Proof. The natural simplicial set map $j : X \to \mathrm{Ex}^\infty X$ consists, in part, of a natural bijection

$$X_0 \xrightarrow{\cong} \mathrm{Ex}^\infty X_0$$

of vertices for all simplicial sets X, and the horizontal arrows in the natural pullback diagrams

$$\begin{tikzcd} \pi_n X \arrow[r] \arrow[d] & \pi_n \mathrm{Ex}^\infty X \arrow[d] \\ X_0 \arrow[r] & \mathrm{Ex}^\infty X_0 \end{tikzcd}$$

are isomorphisms. It follows that the diagram

$$\begin{tikzcd} \tilde{\pi}_n X \arrow[r] \arrow[d] & \tilde{\pi}_n Y \arrow[d] \\ \tilde{X}_0 \arrow[r] & \tilde{Y}_0 \end{tikzcd}$$

is a pullback if and only if the diagram

$$\begin{tikzcd} \tilde{\pi}_n \mathrm{Ex}^\infty X \arrow[r] \arrow[d] & \tilde{\pi}_n \mathrm{Ex}^\infty Y \arrow[d] \\ \tilde{X}_0 \arrow[r] & \tilde{Y}_0 \end{tikzcd}$$

is a pullback.

Remark 4.6. The map $j : X \to \mathrm{Ex}^\infty X$ is a sectionwise equivalence, and is therefore a local weak equivalence by Lemma 4.4. We do not yet have a 2 out of 3 lemma for local weak equivalences (**CM2**) so the trick in the proof of Lemma 4.5 is required to show that the Ex^∞ functor preserves local weak equivalences. This situation is repaired in Lemma 4.27 below.

4.2 Local fibrations

Suppose that $i : K \subset L$ is a cofibration of finite simplicial sets and that $f : X \to Y$ is a map of simplicial presheaves. We say that f has the *local right lifting property* with respect to i if for every diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & X(U) \arrow[d, "f"] \\ L \arrow[r] & Y(U) \end{tikzcd}$$

there is a covering sieve $R \subset \hom(\ ,U)$ such that the lift exists in the diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & X(U) \arrow[r, "\phi^*"] & X(V) \arrow[d, "f"] \\ L \arrow[r] \arrow[urr, dotted] & Y(U) \arrow[r, "\phi^*"'] & Y(V) \end{tikzcd}$$

for every $\phi : V \to U$ in R.

Remark 4.7. There is no requirement for consistency between the lifts along the various members of the sieve R. Thus, if R is generated by a covering family $\phi_i : V_i \to U$, we just require liftings

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & X(U) \arrow[r, "\phi_i^*"] & X(V_i) \arrow[d, "f"] \\ L \arrow[r] \arrow[urr, dotted] & Y(U) \arrow[r, "\phi_i^*"'] & Y(V_i) \end{tikzcd}$$

Exercise 4.8. 1) Suppose given simplicial presheaf maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

such that f and g have the local right lifting property with respect to $i : K \subset L$. Show that the composite $g \cdot f$ has the local right lifting property with respect to the map i.

2) Suppose given a pullback diagram

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d, "f_*"'] & X \arrow[d, "f"] \\ Z \arrow[r] & Y \end{tikzcd}$$

such that f has the local right lifting property with respect to $i : K \subset L$. Show that f_* has the local right lifting property with respect to i.

One summarizes by saying that the class of simplicial presheaf maps having the local right lifting property with respect to $i : K \subset L$ is closed under composition and base change.

Write X^K for the presheaf defined by the function complexes

$$X^K(U) = \mathbf{hom}(K, X(U))$$

Lemma 4.9. *A map $f : X \to Y$ has the local right lifting property with respect to $i : K \to L$ if and only if the simplicial presheaf map*

$$X^L \xrightarrow{(i^*, f_*)} X^K \times_{Y^K} Y^L$$

is a local epimorphism in degree 0.

Proof. The proof is an exercise.

The condition on the map $f : X \to Y$ of Lemma 4.9 is the requirement that the presheaf map

$$\hom(L,X) \xrightarrow{(i^*,f_*)} \hom(K,X) \times_{\hom(K,Y)} \hom(L,Y) \tag{4.6}$$

is a local epimorphism, where $\hom(K,X)$ is the presheaf which is specified in sections by

$$\hom(K,X)(U) = \hom(K,X(U)),$$

or the simplicial set morphisms $K \to X(U)$.

Lemma 4.10. *Suppose that $f : X \to Y$ is a map of simplicial sheaves on $\mathscr{C}$ which has the local right lifting property with respect to an inclusion $i : K \subset L$ of finite simplicial sets, and suppose that $p : \mathbf{Shv}(\mathscr{D}) \to \mathbf{Shv}(\mathscr{C})$ is a geometric morphism. Then the induced map $p^* : p^*X \to p^*Y$ has the local right lifting property with respect to $i : K \subset L$.*

Proof. The identifications

$$p^* \hom(\Delta^n, X) \cong p^*X_n \cong \hom(\Delta^n, p^*X)$$

are natural in simplices Δ^n and simplicial sheaves X, and therefore induce a natural map

$$p^* \hom(K,X) \to \hom(K, p^*X)$$

This map is an isomorphism for all simplicial sheaves X and all finite simplicial sets K, since p^* preserves finite limits.

The map (4.6) is a sheaf epimorphism, since $f : X \to Y$ has the local right lifting property with respect to i. The inverse image functor p^* preserves sheaf epimorphisms, so applying p^* to the map (4.6) gives a sheaf epimorphism which is isomorphic to the map

$$\hom(L, p^*X) \xrightarrow{(i^*,p^*f_*)} \hom(K, p^*X) \times_{\hom(K,p^*Y)} \hom(L, p^*Y).$$

Lemma 4.11. *A simplicial presheaf map $f : X \to Y$ has the local right lifting property with respect to an inclusion $i : K \subset L$ of finite simplicial sets if and only if the induced map $f_* : \tilde{X} \to \tilde{Y}$ of associated simplicial sheaves has the local right lifting property with respect to i.*

Proof. The presheaf map (4.6) is a local epimorphism if and only if the induced map

$$\hom(L,\tilde{X}) \to \hom(K,\tilde{X}) \times_{\hom(K,\tilde{Y})} \hom(L,\tilde{Y})$$

of associated sheaves is a local epimorphism (ie. an epimorphism of sheaves), by Lemma 2.16. Recall that the associated sheaf functor is the inverse image functor for a geometric morphism — see Example 2.25.

Definition 4.12. A *local fibration* is a map which has the local right lifting property with respect to all inclusions $\Lambda_k^n \subset \Delta^n$ of horns in simplices. A simplicial presheaf X is *locally fibrant* if the map $X \to *$ is a local fibration.

Example 4.13. Every sectionwise fibration is a local fibration.

Corollary 4.14. *1) Suppose that $p : \mathbf{Shv}(\mathscr{D}) \to \mathbf{Shv}(\mathscr{C})$ is a geometric morphism. Then the inverse image functor p^* preserves local fibrations.*
2) The associated sheaf functor preserves and reflects local fibrations.

Say that a map $p : X \to Y$ which has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$ is a *local trivial fibration*. Such a map is also called a *hypercover* . This is the natural generalization, to simplicial presheaves, of the concept of a hypercover of a scheme for the étale topology which was introduced by Artin and Mazur [3].

Suppose that X is a simplicial sheaf. Then the map $X \to *$ is a hypercover (or local trivial fibration) if the maps

$$\begin{aligned} &X_0 \to *, \\ &\hom(\Delta^n, X) \to \hom(\partial\Delta^n, X), \ n \geq 1, \end{aligned} \tag{4.7}$$

are sheaf epimorphisms. There is a standard definition

$$\mathrm{cosk}_m(X)_n = \hom(\mathrm{sk}_m \Delta^n, X)$$

and $\mathrm{sk}_{n-1}\Delta^n = \partial\Delta^n$, so that the second map of (4.7) can be written as

$$X_n \to \mathrm{cosk}_{n-1}(X)_n,$$

which is the way that it is displayed in [3].

Example 4.15. Every map which is a sectionwise fibration and a sectionwise weak equivalence is a local trivial fibration.

Corollary 4.16. *1) Suppose that $p : \mathbf{Shv}(\mathscr{D}) \to \mathbf{Shv}(\mathscr{C})$ is a geometric morphism. Then the inverse image functor p^* preserves local trivial fibrations.*
2) The associated sheaf functor preserves and reflects local trivial fibrations.

Corollary 4.17. *The maps $\nu : X \to LX$ and $\eta : X \to L^2X$ are local trivial fibrations.*

Proof. Both maps induce isomorphisms of associated simplicial sheaves, and every isomorphism is plainly a local trivial fibration.

Example 4.18. Suppose that $f : X \to Y$ is a function. There is a groupoid $C(f)$ whose objects are the elements x of X, and whose morphisms are the pairs (x_1, x_2) such that $f(x_1) = f(x_2)$. The set of path components

$$\pi_0 C(f) = \pi_0 BC(f)$$

of the groupoid (and of its associated nerve) is isomorphic to the image $f(X)$ of f, and there is a trivial Kan fibration $BC(f) \to f(X)$ which is natural in functions f. Note that the nerve $BC(f)$ of the groupoid $C(f)$ and constant simplicial set $f(X)$ are both Kan complexes.

It follows that if $f : X \to Y$ is a map of presheaves, then there is a sectionwise trivial fibration $BC(f) \to f(X)$, where $B(C(f))(U)$ is the nerve of the groupoid associated to the function $f : X(U) \to Y(U)$. If the map f is a local epimorphism, then the inclusion $f(X) \subset Y$ induces an isomorphism of associated sheaves, and is therefore a local trivial fibration of constant simplicial presheaves. The groupoid $C(f)$ is the *Čech groupoid* for the map f.

It follows that the canonical map $BC(f) \to Y$ is a local trivial fibration if the presheaf map f is a local epimorphism.

Write $\check{C}(U) = BC(U)$ for the copy of $BC(t)$ associated to the presheaf map $t : U \to *$, where $*$ is the terminal object. If t is a local epimorphism, then the map $\check{C}(U) \to *$ is a local trivial fibration (and therefore a hypercover). This is the *Čech resolution* of the terminal object which is associated to the covering $U \to *$.

1) If $\mathscr{C}$ is the site $op|_T$ of open subsets of a topological space T, and $U_\alpha \subset T$ is an open cover, then the subspaces U_α represent sheaves having the same names, and the map $U = \sqcup_\alpha U_\alpha \to *$ is a sheaf epimorphism. The n-fold product $U^{\times n}$ has the form

$$U^{\times n} = \bigsqcup_{(\alpha_1,\dots,\alpha_n)} U_{\alpha_1} \cap \dots \cap U_{\alpha_n},$$

and so the simplicial sheaf $\check{C}(U)$ is represented by a simplicial space, which is the classical Čech resolution associated to the covering $U_\alpha \subset T$.

2) If $\mathscr{C}$ is the étale site $et|_k$ of a field k, and L/k is a finite Galois extension with Galois group G, then the scheme homomorphism $\mathrm{Sp}(L) \to \mathrm{Sp}(k)$ represents a covering $\mathrm{Sp}(L) \to *$ on $et|_k$. There is a canonical sheaf isomorphism

$$G \times \mathrm{Sp}(L) \xrightarrow{\cong} \mathrm{Sp}(L) \times \mathrm{Sp}(L)$$

by elementary Galois theory, and the sheaf of groupoids underlying $C(\mathrm{Sp}(L))$ is isomorphic to the translation groupoid $E_G \mathrm{Sp}(L)$ for the action of G on $\mathrm{Sp}(L)$. It follows that the Čech resolution $\check{C}(L) := \check{C}(\mathrm{Sp}(L))$ is isomorphic in simplicial sheaves to the Borel construction $EG \times_G \mathrm{Sp}(L)$ for the action of the Galois group G on $\mathrm{Sp}(L)$.

Such an observation holds, more generally, for all principal bundles (torsors) in sheaf categories. This will be discussed in much more detail in Chapter 8.

Lemma 4.19. *Suppose that X and Y are presheaves of Kan complexes. Then a map $p : X \to Y$ is a local fibration and a local weak equivalence if and only if p is a local trivial fibration.*

It will be shown (Theorem 4.32) that an arbitrary map $p : X \to Y$ of simplicial presheaves is a local weak equivalence and a local fibration if and only if it is a local trivial fibration.

Example 4.20. If $f : X \to Y$ is a local epimorphism of presheaves, then the local trivial $BC(f) \to Y$ of Example 4.18 is a local weak equivalence. In particular, every Čech resolution $\check{C}(U) \to *$ associated to a covering $U \to *$ is a local weak equivalence.

Proof (Proof of Lemma 4.19). Suppose that p is a local fibration and a local weak equivalence, and that we have a commutative diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X(U) \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y(U) \end{tikzcd} \tag{4.8}$$

of simplicial set maps. The idea is to show that this diagram is locally homotopic to diagrams

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X(V) \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y(V) \end{tikzcd}$$

for which the lift exists. This means that there are homotopies

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \arrow[r] \arrow[d] & X(V) \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[r] & Y(V) \end{tikzcd}$$

from the diagrams

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X(U) \arrow[r, "\phi^*"] & X(V) \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y(U) \arrow[r, "\phi^*"'] & Y(V) \end{tikzcd}$$

to the corresponding diagrams above for all $\phi : V \to U$ in a covering for U. If such local homotopies exist, then solutions to the lifting problems

$$\begin{tikzcd} (\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \arrow[r] \arrow[d] & X(V) \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[r] & Y(V) \end{tikzcd}$$

have local solutions for each V, and so the original lifting problem is solved on a refined covering of U.

The required local homotopies are created by arguments similar to the proof of the corresponding result in the simplicial set case [24, I.7.10]. Here are the steps in the construction:

a) The diagram (4.8) is homotopic to a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "{(\alpha_0,x,\dots,x)}"] \arrow[d] & X(U) \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y(U) \end{tikzcd} \tag{4.9}$$

for some choice of base point $x \in X(U)$, since X and Y are presheaves of Kan complexes.

b) The element $[\alpha_0] \in \pi_{n-1}(X(U),x)$ vanishes locally in X since its image vanishes in $\pi_{n-1}(Y(U),p(x))$, so that the diagram (4.9) is locally homotopic to a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "x"] \arrow[d] & X(V) \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"'] & Y(V) \end{tikzcd} \tag{4.10}$$

c) The element $[\beta] \in \pi_n(Y(V),p(x))$ lifts locally to X, and so the diagram (4.10) is locally homotopic to a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "x"] \arrow[d] & X(W) \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"'] \arrow[ur] & Y(W) \end{tikzcd}$$

for which the indicated lifting exists.

For the converse, show that the induced presheaf maps

$$\pi_0 X \to \pi_0 Y,$$
$$\pi_i(X|_U,x) \to \pi_i(Y|_U,p(x))$$

are local epimorphisms and local monomorphisms — use presheaves of simplicial homotopy groups for this.

Lemma 4.21. *Suppose that a simplicial presheaf map $f : X \to Y$ is a local trivial fibration. Then f is a local fibration and a local weak equivalence.*

Proof. The local fibration part of the claim is easy, since the map f has the right lifting property with respect to all inclusions of finite simplicial sets.

The induced map

$$f : \mathrm{Ex}(X) \to \mathrm{Ex}(Y)$$

has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$, since f has the local right lifting property with respect to all $\mathrm{sd}\,\partial\Delta^n \to \mathrm{sd}\,\Delta^n$. It follows that the map

$$f : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(Y)$$

has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$ and is a map of presheaves of Kan complexes. Finish by using Lemma 4.9 and Lemma 4.19.

Corollary 4.22. *The maps $\nu : X \to LX$ and $\eta : X \to L^2X$ are local fibrations and local weak equivalences.*

Proof. This is a consequence of Corollary 4.17 and Lemma 4.21.

Corollary 4.23. *1) A map $f : X \to Y$ of simplicial presheaves is a local weak equivalence if and only if the induced map $f_* : LX \to LY$ is a local weak equivalence.*
2) A map $f : X \to Y$ of simplicial presheaves is a local weak equivalence if and only if the induced map $f_* : \tilde{X} \to \tilde{Y}$ of associated sheaves is a local weak equivalence.
3) A map $f : X \to Y$ of simplicial presheaves is a local weak equivalence if and only if the induced map $f_* : L^2\,\mathrm{Ex}^\infty X \to L^2\,\mathrm{Ex}^\infty Y$ is a local weak equivalence.

Proof. For statement 1), the map $\eta : X \to LX$ is a local weak equivalence, so that the induced diagram of sheaves

$$\begin{tikzcd}
\tilde{\pi}_n X \arrow[r] \arrow[d] & \tilde{\pi}_n LX \arrow[d] \\
\tilde{X}_0 \arrow[r] & \widetilde{LX}_0
\end{tikzcd}$$

is a pullback. The map $\tilde{X}_0 \to \widetilde{LX}_0$ is an isomorphism, so that both horizontal arrows in the diagram are isomorphisms of sheaves. Finish with the argument for Lemma 4.5.

For statement 2), recall that $\tilde{X} = L^2X$, and use statement 1). Statement 3) is an easy consequence of statement 2) and Lemma 4.9.

4.3 First applications of Boolean localization

The local weak equivalence and local fibration concepts for simplicial presheaves have very special interpretations for simplicial sheaves on a complete Boolean algebra.

Lemma 4.24. *Suppose that $\mathscr{B}$ is a complete Boolean algebra.*

1) A map $p : X \to Y$ of simplicial sheaves on $\mathscr{B}$ is a local (respectively local trivial) fibration if and only if all maps $p : X(b) \to Y(b)$ are Kan fibrations (respectively trivial Kan fibrations) for all $b \in \mathscr{B}$.

2) *A map $f : X \to Y$ of locally fibrant simplicial sheaves on $\mathscr{B}$ is a local weak equivalence if and only if all maps $f : X(b) \to Y(b)$ are weak equivalences of simplicial sets for all $b \in \mathscr{B}$.*

Proof. The induced map

$$X^{\Delta^n} \to Y^{\Delta^n} \times_{Y^{\partial\Delta^n}} X^{\partial\Delta^n}$$

is a sheaf epimorphism in degree 0 if and only if it is a sectionwise epimorphism in degree 0, since $\mathbf{Shv}(\mathscr{B})$ satisfies the Axiom of Choice (Lemma 2.30). The local fibration statement is similar.

For part 2), suppose that f is a local weak equivalence. The map f has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & X \times_Y Y^{\Delta^1} \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where p is a sectionwise Kan fibration and j is right inverse to a sectionwise trivial Kan fibration — this is by a standard construction, but see Section 5.1 below. All objects in the diagram are sheaves of Kan complexes. The map p is a local weak equivalence and a local fibration, and is therefore a sectionwise weak equivalence by Lemma 4.19 and part 1). It follows that f is a sectionwise weak equivalence.

The converse follows from Lemma 4.4.

Lemma 4.25. *Suppose that the geometric morphism*

$$p : \mathbf{Shv}(\mathscr{B}) \to \mathbf{Shv}(\mathscr{C})$$

is a Boolean localization. A map $f : X \to Y$ of simplicial sheaves on $\mathscr{C}$ is a local trivial fibration (respectively local fibration) if and only if the induced map

$$p^*f : p^*X \to p^*Y$$

is a sectionwise trivial Kan fibration (respectively sectionwise Kan fibration) in $s\mathbf{Shv}(\mathscr{B})$.

Proof. The simplicial sheaf map

$$X^{\Delta^n} \to X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}$$

is a sheaf epimorphism in degree zero if and only if the induced map

$$p^*X^{\Delta^n} \to p^*X^{\partial\Delta^n} \times_{p^*Y^{\partial\Delta^n}} p^*Y^{\Delta^n}$$

is a sheaf epimorphism in degree 0, by Lemma 2.26. Now use Lemma 4.24.

Proposition 4.26. *Suppose that the geometric morphism*

$$p : \mathbf{Shv}(\mathscr{B}) \to \mathbf{Shv}(\mathscr{C})$$

is a Boolean localization, and that $f : X \to Y$ is a map of $s\mathbf{Pre}(\mathscr{C})$. Then f is a local weak equivalence if and only if the induced map

$$f_* : p^*\tilde{X} \to p^*\tilde{Y}$$

is a local weak equivalence of $s\mathbf{Shv}(\mathscr{B})$.

Proof. Suppose, first of all, that X and Y are presheaves of Kan complexes. Then f is a weak equivalence if and only if the following conditions are satisfied:

a) the induced map $\tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ of sheaves of path components is an isomorphism
2) the diagram

$$\begin{array}{ccc} \tilde{\pi}_0 F_n(X) & \longrightarrow & \tilde{\pi}_0 F_n(Y) \\ \downarrow & & \downarrow \\ \tilde{X}_0 & \longrightarrow & \tilde{Y}_0 \end{array} \tag{4.11}$$

a pullback diagram of sheaves, for all $n \geq 1$.

Recall that the canonical map $\pi_0 F_n(X) \to X_0$ is defined by applying the path component functor to the map $F_n(X) \to X_0$ in the pullback diagram

$$\begin{array}{ccc} F_n(X) & \longrightarrow & \mathbf{hom}(\Delta^n, X) \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & \mathbf{hom}(\partial\Delta^n, X) \end{array}$$

where i^* is the sectionwise functor which is defined by precomposition with the inclusion $i : \partial\Delta^n \subset \Delta^n$.

These constructions commute with inverse image (and sheafification) up to natural isomorphism. It follows that the diagram (4.11) is isomorphic with the diagram

$$\begin{array}{ccc} \tilde{\pi}_0 F_n(\tilde{X}) & \longrightarrow & \tilde{\pi}_0 F_n(\tilde{Y}) \\ \downarrow & & \downarrow \\ \tilde{X}_0 & \longrightarrow & \tilde{Y}_0 \end{array} \tag{4.12}$$

and that the inverse image of diagram (4.12) under p^* is isomorphic with the diagram

$$\begin{array}{ccc} \tilde{\pi}_0 F_n(p^*\tilde{X}) & \longrightarrow & \tilde{\pi}_0 F_n(p^*\tilde{Y}) \\ \downarrow & & \downarrow \\ p^*\tilde{X}_0 & \longrightarrow & p^*\tilde{Y}_0 \end{array} \tag{4.13}$$

The objects $p^*\tilde{X}$ and $p^*\tilde{Y}$ are sheaves of Kan complexes, and so the map $p^*\tilde{X} \to p^*\tilde{Y}$ is a local weak equivalence if and only if the map $\tilde{\pi}_0 p^*\tilde{X} \to \tilde{\pi}_0 p^*\tilde{Y}$ is a sheaf isomorphism and all diagrams (4.13) are pullbacks. But this is true if and only if the map $\tilde{\pi}_0\tilde{X} \to \tilde{\pi}_0\tilde{Y}$ is a sheaf isomorphism on $\mathscr{C}$ and all diagrams (4.12) are pullbacks, since p^* preserves and reflects pullbacks. It follows that $f : X \to Y$ is a local weak equivalence of presheaves of Kan complexes on $\mathscr{C}$ if and only if the map $p^*\tilde{X} \to p^*\tilde{Y}$ is a local (hence sectionwise) weak equivalence on $\mathscr{B}$.

In general, we know that $f : X \to Y$ is a local weak equivalence of simplicial presheaves if and only if the map $\mathrm{Ex}^\infty X \to \mathrm{Ex}^\infty$ is a local weak equivalence. We also know that this is true if and only if the map

$$p^*L^2\,\mathrm{Ex}^\infty X \to p^*L^2\,\mathrm{Ex}^\infty Y$$

is a local weak equivalence of simplicial sheaves on $\mathscr{B}$. By exactness of p^* and the associated sheaf functor L^2, there is a natural isomorphism

$$p^*L^2\,\mathrm{Ex}^\infty X \cong L^2\,\mathrm{Ex}^\infty\, p^*\tilde{X}$$

for simplicial presheaves X. It follows that $f : X \to Y$ is a local weak equivalence of simplicial sheaves on $\mathscr{C}$ if and only if $f_* : p^*\tilde{X} \to p^*\tilde{Y}$ is a local weak equivalence of simplicial sheaves on $\mathscr{B}$.

We can now give a proof of the Quillen's axiom **CM1** for local weak equivalences.

Lemma 4.27. *Suppose given a commutative diagram of simplicial presheaf maps*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \tag{4.14}$$

on a Grothendieck site $\mathscr{C}$. If any two of f, g or h are local weak equivalences then so is the third.

Proof. Suppose that $p : \mathbf{Shv}(\mathscr{B}) \to \mathbf{Shv}(\mathscr{C})$ is a Boolean localization. Then a simplicial presheaf map $f : X \to Y$ is a local weak equivalence if and only if the induced map

$$f_* : p^*L^2\,\mathrm{Ex}^\infty X \to p^*L^2\,\mathrm{Ex}^\infty Y$$

is a sectionwise equivalence of sheaves of Kan complexes on $\mathscr{B}$. Apply the functor $p^*L^2\,\mathrm{Ex}^\infty$ to the triangle (4.14) to prove the result.

It is a consequence of Proposition 1.23 that the category $\mathbf{Pre}(\mathscr{C})$ of simplicial presheaves on $\mathscr{C}$ has a projective model structure, for which the fibrations and weak equivalences are defined sectionwise, and the cofibrations are specified by a left lifting property. Recall that a *projective cofibration* is a map which has the left lifting property with respect to all sectionwise trivial fibrations.

Observe that the natural map $j : X \to \mathrm{Ex}^\infty X$ is a sectionwise fibrant model for a simplicial presheaf X, in the sense that j is a sectionwise weak equivalence and the simplicial presheaf $\mathrm{Ex}^\infty X$ is sectionwise fibrant. Every map of simplicial presheaves $f : X \to Y$ can be replaced up to sectionwise weak equivalence by a map of sectionwise fibrant objects, in the sense that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\simeq"'] & Y \arrow[d, "\simeq"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

where the vertical maps are sectionwise equivalences (even trivial projective cofibrations) and $f' : X' \to Y'$ is a map between sectionwise fibrant simplicial presheaves (or presheaves of Kan complexes). On account of Lemma 4.27 (and Lemma 4.4), the map f is a local weak equivalence if and only if the map $f' : X' \to Y'$ of sectionwise fibrant models is a local weak equivalence, and this is so if and only if the induced map

$$f'_* : p^*\tilde{X}' \to p^*\tilde{Y}'$$

is a sectionwise equivalence of sheaves of Kan complexes for some Boolean localization $p : \mathbf{Shv}(\mathscr{B}) \to \mathbf{Shv}(\mathscr{C})$, by Proposition 4.26, Lemma 4.24 and Corollary 4.14.

Say that a simplicial presheaf map $i : A \to B$ is a *cofibration* if it is a monomorphism in all sections. It follows from the proof of Proposition 1.23 that every projective cofibration is a cofibration.

Lemma 4.28. *Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\ B \arrow[r] & D \end{tikzcd} \tag{4.15}$$

of simplicial sheaves on a complete Boolean algebra $\mathscr{B}$ such that i is a cofibration and a local weak equivalence. Then the map i_* is a cofibration and a local weak equivalence.

Proof. Form a diagram

$$\begin{tikzcd} B \arrow[d, "\simeq"'] & A \arrow[l, "i"'] \arrow[r, "\alpha"] \arrow[d, "\simeq"] & C \arrow[d] \\ B' & A' \arrow[l, "i'"] \arrow[r, "\alpha'"'] & C' \end{tikzcd}$$

in which the vertical maps are sectionwise weak equivalences, i' is a cofibration, and the objects A', B' and C' are sectionwise fibrant, and form the pushout

$$\begin{tikzcd} A' \arrow[r, "\alpha'"] \arrow[d, "i'"'] & C' \arrow[d, "i'_*"] \\ B' \arrow[r] & D' \end{tikzcd}$$

all in the simplicial presheaf category on $\mathscr{B}$. The induced map $D \to D'$ is a sectionwise weak equivalence by properness of the projective model structure, and it follows that i_* is a local weak equivalence if and only if i'_* is a local weak equivalence.

Sheafifying gives a pushout diagram of simplicial sheaves

$$\begin{tikzcd} \tilde{A}' \arrow[r] \arrow[d, "\tilde{i}'"'] & \tilde{C}' \arrow[d] \\ \tilde{B}' \arrow[r] & \tilde{D}' \end{tikzcd}$$

which is locally equivalent to the original, and for which $\tilde{i}'$ is a cofibration. We can therefore assume that the objects A, B and C in the diagram (4.15) are locally fibrant.

The map $i : A \to B$ is a sectionwise weak equivalence, by Lemma 4.24. Sectionwise trivial cofibrations are closed under pushout in the simplicial presheaf category, and since D is the associated sheaf of the presheaf pushout, the map $i_* : C \to D$ must then be a local weak equivalence by Lemma 4.27.

Corollary 4.29. *Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\ B \arrow[r] & D \end{tikzcd}$$

of simplicial presheaves on a Grothendieck site $\mathscr{C}$, and suppose that i is a cofibration and a local weak equivalence. Then the map i_* is a local weak equivalence.

Proof. Suppose that $p : \mathbf{Shv}(\mathscr{B}) \to \mathbf{Shv}(\mathscr{C})$ is a Boolean localization. The functor p^*L^2 preserves cofibrations and pushouts, and preserves and reflects local weak equivalences.

The map $p^*\tilde{A} \to p^*\tilde{B}$ induced by i is a local weak equivalence and a cofibration, so the map $p^*\tilde{C} \to p^*\tilde{D}$ induced by i_* is a local weak equivalence by Lemma 4.28. It follows from Proposition 4.26 that i_* is a local weak equivalence.

Lemma 4.30. *1) Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \\ B \arrow[r, "f_*"'] & D \end{tikzcd}$$

in simplicial presheaves on $\mathscr{C}$ such that i is a cofibration and f is a local weak equivalence. Then the map f_* is a local weak equivalence.

2) Suppose given a pullback diagram

$$\begin{tikzcd} W \arrow[r, "g_*"] \arrow[d] & X \arrow[d, "p"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

in $s\mathbf{Pre}(\mathscr{C})$ such that p is a local fibration and g is a local weak equivalence. Then the map g_* is a local weak equivalence.

Proof. Suppose that $p : \mathbf{Shv}(\mathscr{B}) \to \mathbf{Shv}(\mathscr{C})$ is a Boolean localization. Since the inverse image functor p^* reflects local weak equivalences of simplicial sheaves by Proposition 4.26, it suffices (by Corollary 4.22) to assume that the diagrams in statements 1) and 2) are diagrams of simplicial sheaves on $\mathscr{B}$.

The proof of statement 1) then uses the method of proof of Lemma 4.28: we can assume that the simplicial sheaves A, B and C are locally fibrant up to local weak equivalence, so that f is a sectionwise equivalence. It follows that f_* is a sectionwise weak equivalence.

For the proof of statement 2), we can assume that the simplicial sheaves Z, Y and X are locally fibrant, by a sectionwise fibrant replacement argument at the presheaf level. Then g is a sectionwise weak equivalence and p is a sectionwise fibration, both by Lemma 4.24, so that the map g_* is a sectionwise weak equivalence by properness of the projective model structure for simplicial presheaves.

Lemma 4.31. *Suppose that $\mathscr{B}$ is a complete Boolean algebra. Suppose that $p : X \to Y$ is a map of simplicial sheaves on $\mathscr{B}$ such that p is a sectionwise Kan fibration and a local weak equivalence. Then p is a sectionwise trivial fibration.*

Proof. The functor $X \mapsto L^2 \operatorname{Ex}^\infty X$ preserves sectionwise Kan fibrations and preserves pullbacks. Also the sectionwise fibration $p : X \to Y$ is local weak equivalence if and only if the induced map $p_* : L^2 \operatorname{Ex}^\infty X \to L^2 \operatorname{Ex}^\infty Y$ is a sectionwise weak equivalence. It follows that the family of all maps of simplicial sheaves on $\mathscr{B}$ which are simultaneously sectionwise Kan fibrations and local weak equivalences is closed under pullback.

Suppose given a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d, "i"'] & X(b) \arrow[d, "p"] \\ \Delta^n \arrow[r, "\beta"'] & Y(b) \end{tikzcd}$$

The simplex Δ^n contracts onto the vertex 0; write $h : \Delta^n \times \Delta^1 \to \Delta^n$ for the contracting homotopy. Let $h' : \partial\Delta^n \times \Delta^1 \to X(b)$ be a choice of lifting in the diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d] & X(b) \arrow[d, "p"] \\
\partial\Delta^n \times \Delta^1 \arrow[ur, "h'"] \arrow[r, "\beta\cdot h\cdot(i\times 1)"'] & Y(b)
\end{tikzcd}$$

Then the original diagram is homotopic to a diagram of the form

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha'"] \arrow[d, "i"'] & X(b) \arrow[d, "p"] \\
\Delta^n \arrow[r, "x"'] & Y(b)
\end{tikzcd}$$

where $x : \Delta^n \to Y(b)$ is constant at the vertex $x \in Y(b)$. Consider the induced diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, "i"'] & (L_b\Delta^0 \times_Y X)(b) \arrow[d, "p_*"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & L_b\Delta^0(b)
\end{tikzcd}$$

where L_b is the left adjoint of the b-sections functor $X \mapsto X(b)$ in sheaves. The object $L_b\Delta^0$ is the sheaf associated to a diagram of points and is therefore locally fibrant, and is thus a sheaf of Kan complexes. The map $p_* : L_b\Delta^0 \times_Y X \to L_b\Delta^0$ is a local fibration and a local weak equivalence between sheaves of Kan complexes and is therefore a sectionwise trivial fibration by Lemma 4.24, so the indicated lift exists.

Generally, if U is an object of a Grothendieck site $\mathscr{C}$, then the left adjoint L_U of the U-sections functor $X \mapsto X(U)$ can be defined for simplicial sets K by

$$L_U(K) = K \times \hom(\ ,U).$$

It is an exercise to show that L_U preserves cofibrations, takes weak equivalences (respectively fibrations) to sectionwise weak equivalences (respectively sectionwise fibrations).

The following result is now a corollary of Lemma 4.31:

Theorem 4.32. *A map $q : X \to Y$ of simplicial presheaves on $\mathscr{C}$ is a local weak equivalence and a local fibration if and only if it has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n, n \geq 0$.*

To paraphrase, this result says that a map is a local fibration and a local weak equivalence if and only if it is a local trivial fibration.

Proof. If q has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$ then it is a local fibration and a local weak equivalence, by Lemma 4.21. We prove the converse statement here.

Suppose that $p : \mathbf{Shv}(\mathscr{B}) \to \mathbf{Shv}(\mathscr{C})$ is a Boolean localization. Then p^*L^2q is a local weak equivalence and a local fibration, and is therefore a sectionwise trivial fibration by Lemma 4.31. The functor p^*L^2 reflects local epimorphisms, so that the map

$$X^{\Delta^n} \to Y^{\Delta^n} \times_{Y^{\partial\Delta^n}} X^{\partial\Delta^n}$$

is a local epimorphism in degree 0.

Chapter 5
Model structures

5.1 The injective model structure

We begin by reviewing the classical fibration replacement construction from simplicial homotopy theory.

Suppose that $f : X \to Y$ is a map of Kan complexes, and form the diagram

$$\begin{tikzcd}
X \times_Y Y^I \arrow[r, "f_*"] \arrow[d, "d_{0*}"'] & Y^I \arrow[r, "d_1"] \arrow[d, "d_0"] & Y \\
X \arrow[r, "f"'] \arrow[ur, "sf"] & Y &
\end{tikzcd}$$

in which the square is a pullback. Then d_0 is a trivial fibration since Y is a Kan complex, so d_{0*} is a trivial fibration. The section s of d_0 (and d_1) induces a section s_* of d_{0*}, and

$$(d_1 f_*)s_* = d_1(sf) = f$$

Finally, there is a pullback diagram

$$\begin{tikzcd}
X \times_Y Y^I \arrow[r, "f_*"] \arrow[d, "(d_{0*}, d_1 f_*)"'] & Y^I \arrow[d, "(d_0, d_1)"] \\
X \times Y \arrow[r, "f \times 1"'] & Y \times Y
\end{tikzcd}$$

and the projection map $pr_R : X \times Y \to Y$ is a fibration since X is a Kan complex, so that $pr_R(d_{0*}, d_1 f_*) = d_1 f_*$ is a fibration.

Write $Z_f = X \times_Y Y^I$ and $\pi_f = d_1 f_*$. Then we have functorial replacement

$$\begin{tikzcd}
X \arrow[r, "s_*"] \arrow[dr, "f"'] & Z_f \arrow[r, "d_{0*}"] \arrow[d, "\pi"] & X \\
& Y &
\end{tikzcd} \tag{5.1}$$

of f by a fibration π, where d_{0*} is a trivial fibration such that $d_{0*}s_* = 1$.

The same argument can be repeated exactly within the theory of local fibrations, giving the following:

Lemma 5.1. *Suppose that $f : X \to Y$ is a map between locally fibrant simplicial presheaves.*

1) *The map f has a natural factorization (5.1) for which π is a local fibration, d_{0*} is a local trivial fibration, and $d_{0*}s_* = 1_X$.*
2) *The map f is a local weak equivalence if and only if the map π in the factorization (5.1) is a local trivial fibration.*

The second statement of the Lemma follows from Theorem 4.32.

Suppose again that $f : X \to Y$ is a simplicial set map, and form the diagram

$$\begin{tikzcd}
X \arrow[rr, "j"] \arrow[dr, "\theta_f"] \arrow[dd, "f"'] & & \mathrm{Ex}^\infty X \arrow[dr, "s_*"] \arrow[dd, "f_*"] & \\
& \tilde{Z}_f \arrow[rr, "j_*"'] \arrow[dl, "\pi_f"] & & Z_{f_*} \arrow[dl, "\pi"] \\
Y \arrow[rr, "j"'] & & \mathrm{Ex}^\infty Y &
\end{tikzcd}$$

in which the front face is a pullback. Then π_f is a fibration, and θ_f is a weak equivalence since j_* is a weak equivalence by properness of the model structure for simplicial sets.

The construction taking a map f to the factorization

$$\begin{tikzcd}
X \arrow[r, "\theta_f"] \arrow[dr, "f"'] & \tilde{Z}_f \arrow[d, "\pi_f"] \\
& Y
\end{tikzcd} \tag{5.2}$$

also has the following properties:

a) it is natural in f
b) it preserves filtered colimits in f
c) if X and Y are α-bounded where α is some infinite cardinal, then so is $\tilde{Z}_f$

Say that a simplicial set X is *α-bounded* if $|X_n| < \alpha$ for all $n \geq 0$, or in other words if α is an upper bound for the cardinality of all sets of simplices of X. A simplicial presheaf Y is α-bounded if all of the simplicial sets $Y(U)$, $U \in \mathscr{C}$, are α-bounded.

This construction (5.2) carries over to simplicial presheaves, giving a natural factorization

$$\begin{tikzcd} X \arrow[r, "\theta_f"] \arrow[dr, "f"'] & \tilde{Z}_f \arrow[d, "\pi_f"] \\ & Y \end{tikzcd} \tag{5.3}$$

of a simplicial presheaf map $f : X \to Y$ such that θ_f is a sectionwise weak equivalence and π_f is a sectionwise fibration. Here are some further properties of this factorization:

a) it preserves filtered colimits in f
b) if X and Y are α-bounded where α is some infinite cardinal, then so is $\tilde{Z}_f$
c) f is a local weak equivalence if and only if π_f has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$.

Statement c) is a consequence of Theorem 4.32.

Suppose that $\mathscr{C}$ is a Grothendieck site, and recall that we assume that such a category is small. Suppose that α is an infinite cardinal such that $\alpha > |\mathrm{Mor}(\mathscr{C})|$. Choose another infinite cardinal $\lambda > 2^\alpha$.

The following result is a *bounded cofibration lemma.*

Lemma 5.2. *Suppose that $i : X \to Y$ is a cofibration and a local weak equivalence of $s\mathbf{Pre}(\mathscr{C})$. Suppose that $A \subset Y$ is an α-bounded subobject of Y. Then there is an α-bounded subobject B of Y such that $A \subset B$ and such that the map $B \cap X \to B$ is a local weak equivalence.*

Proof. Write $\pi_B : Z_B \to B$ for the natural pointwise Kan fibration replacement for the cofibration $B \cap X \to B$. The map $\pi_Y : Z_Y \to Y$ has the local right lifting property with respect to all $\partial\Delta^n \subset \Delta^n$.

Supppose given a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & Z_A(U) \arrow[d, "\pi_A"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & A(U) \end{tikzcd}$$

where A is α-bounded. The lifting problem can be solved locally over Y along some covering sieve for U having at most α elements. $Z_Y = \varinjlim_{|B|<\alpha} Z_B$ since Y is a filtered colimit of its α-bounded subobjects. It follows that there is an α-bounded subobject $A' \subset Y$ with $A \subset A'$ such that the original lifting problem can be solved over A'. The list of all such lifting problems is α-bounded, so there is an α-bounded subobject $B_1 \subset Y$ with $A \subset B_1$ so that all lifting problems as above over A can be solved locally over B_1. Repeat this procedure countably many times to produce an ascending family

$$A = B_0 \subset B_1 \subset B_2 \subset \dots$$

of α-bounded subobjects of Y such that all lifting local lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & Z_{B_i}(U) \arrow[d, "\pi_{B_i}"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & B_i(U) \end{tikzcd}$$

over B_i can be solved over B_{i+1}. Set $B = \cup_i B_i$.

Say that a map $p : X \to Y$ of $s\mathbf{Pre}(\mathscr{C})$ is an *injective fibration* if p has the right lifting property with respect to all maps $A \to B$ which are cofibrations and local weak equivalences.

Remark 5.3. Injective fibrations are also called *global fibrations* in the literature, for example in [32]. This use of this name originated in early work of Brown and Gersten [10], but has declined with the introduction of the various model structures associated with motivic homotopy theory. The point of the term "injective fibration" is that the behaviour of an injective fibrant object is roughly analogous to that of an injective object in an abelian category.

Say that a map $A \to B$ of simplicial presheaves is an α*-bounded cofibration* if it is a cofibration and the object B is α-bounded. It follows that A is α-bounded as well.

In this section, a *trivial cofibration* is a map of simplicial presheaves which is a cofibration and a local weak equivalence. This is standard terminology within model structures, and is consistent with the injective model structure which appears in Theorem 5.8 below. Similarly, a *trivial fibration* is a map which is an injective fibration and a local weak equivalence.

Lemma 5.4. *The map $p : X \to Y$ is an injective fibration if and only if it has the right lifting property with respect to all α-bounded trivial cofibrations.*

Proof. Suppose that $p : X \to Y$ has the right lifting property with respect to all α-bounded trivial cofibrations, and suppose given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

where i is a trivial cofibration. Consider the poset of partial lifts

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[dd, "p"] \\ A' \arrow[ur] \arrow[d] & \\ B \arrow[r] & Y \end{tikzcd}$$

in which the maps $A \to A' \to B$ are trivial cofibrations. This poset is non-empty: given $x \in B(U) - A(U)$ there is an α-bounded subcomplex $C \subset B$ with $x \in C(U)$ (let C be the image of the map $L_U\Delta^n \to B$ which is adjoint to the simplex $x : \Delta^n \to B(U)$), and there is an α-bounded subcomplex $C' \subset B$ with $C \subset C'$ and $i_* : C' \cap A \to C'$ a trivial cofibration. Then $x \in C' \cup A$, and there is a diagram

$$\begin{tikzcd}
C' \cap A \arrow[r] \arrow[d, "i_*"'] & A \arrow[r] \arrow[d] & X \arrow[dd, "p"] \\
C' \arrow[r] & C' \cup A \arrow[ur] \arrow[d] & \\
& B \arrow[r] & Y
\end{tikzcd}$$

where the indicated lift exists because p has the right lifting property with respect to the α-bounded trivial cofibration i_*. The map $A \to C' \cup A$ is a trivial cofibration by Corollary 4.29.

The poset of partial lifts has maximal elements by Zorn's Lemma, and the maximal elements of the poset must have the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur] & Y
\end{tikzcd}$$

Recall that one defines

$$L_U K = \hom(\ ,U) \times K$$

for $U \in \mathscr{C}$ and simplicial sets K, and that the functor $K \mapsto L_U K$ is left adjoint to the U-sections functor $X \mapsto X(U)$.

Lemma 5.5. *Suppose that $q : Z \to W$ has the right lifting property with respect to all cofibrations. Then q is an injective fibration and a local weak equivalence.*

Proof. The map q is obviously an injective fibration, and it has the right lifting property with respect to all cofibrations $L_U\partial\Delta^n \to L_U\Delta^n$, so that all maps $q : Z(U) \to W(U)$ are trivial Kan fibrations. It follows that q is a local weak equivalence.

Lemma 5.6. *A map $q : Z \to W$ has the right lifting property with respect to all cofibrations if and only if it has the right lifting property with respect to all α-bounded cofibrations.*

Proof. The proof of this result is an exercise.

Lemma 5.7. *Any simplicial presheaf map $f : X \to Y$ has factorizations*

$$
\begin{tikzcd}
 & Z \arrow[dr, "p"] & \\
X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\
 & W \arrow[ur, "q"'] &
\end{tikzcd}
$$

where

1) the map i is a cofibration and a local weak equivalence, and p is an injective fibration,
2) the map j is a cofibration and p has the right lifting property with respect to all cofibrations (and is therefore an injective fibration and a local weak equivalence)

Proof. For the first factorization, choose a cardinal $\lambda > 2^{\alpha}$ and do a transfinite small object argument of size λ to solve all lifting problems

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}
$$

arising from locally trivial cofibrations i which are α-bounded. We need to know that locally trivial cofibrations are closed under pushout, but this is proved in Corollary 4.29. The small object argument stops on account of the condition on the size of the cardinal λ.

The second factorization is similar, and uses Lemma 5.6.

The main results of this section say that the categories of simplicial presheaves and simplicial sheaves on a Grothendieck site admit well behaved model structures which are Quillen equivalent.

Theorem 5.8. *Suppose that $\mathscr{C}$ is a small Grothendieck site. Then the category $s\mathbf{Pre}(\mathscr{C})$, with local weak equivalences, cofibrations and injective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

Proof. The simplicial presheaf category $s\mathbf{Pre}(\mathscr{C})$ has all small limits and colimits, giving **CM1**. The weak equivalence axiom **CM2** was proved in Lemma 4.27 with a Boolean localization argument. The retract axiom **CM3** is trivial to verify — use the pullback description of local weak equivalences to see the weak equivalence part. The factorization axiom **CM5** is Lemma 5.7.

Suppose that $\pi : X \to Y$ is an injective fibration and a local weak equivalence. Then by the proof of Lemma 5.7, π has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\pi"'] & W \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where p has the right lifting property with respect to all cofibrations and is therefore a local weak equivalence. Then j is a local weak equivalence, and so π is a retract of p. Thus π has the right lifting property with respect to all cofibrations, giving **CM4**.

The simplicial model structure comes from the *function complex* $\mathbf{hom}(X,Y)$, with

$$\mathbf{hom}(X,Y)_n = \hom_{s\mathbf{Pre}(\mathscr{C})}(X \times \Delta^n, Y).$$

One shows that if $i : A \to B$ is a cofibration of simplicial presheaves and $j : K \to L$ is a cofibration of simplicial sets, then the induced map

$$(B \times K) \cup (A \times L) \to B \times L$$

is a cofibration which is a local weak equivalence if either i is a local weak equivalence of simplicial presheaves or j is a weak equivalence of simplicial sets.

The properness of the model structure follows from Lemma 4.30.

It is a consequence of the proof of the model axioms that a generating set I for the class of trivial cofibrations is given by the set of all α-bounded trivial cofibrations, while the set J of α-bounded cofibrations generates the class of cofibrations.

Write $s\mathbf{Shv}(\mathscr{C})$ for the category of simplicial sheaves on $\mathscr{C}$. Say that a map $f : X \to Y$ is a *local weak equivalence* of simplicial sheaves if it is a local weak equivalence of simplicial presheaves. A *cofibration* of simplicial sheaves is a monomorphism, and an *injective fibration* is a map which has the right lifting property with respect to all trivial cofibrations.

Theorem 5.9. *Let $\mathscr{C}$ be a small Grothendieck site.*

1) The category $s\mathbf{Shv}(\mathscr{C})$ with local weak equivalences, cofibrations and injective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.

2) The inclusion i of sheaves in presheaves and the associated sheaf functor L^2 together induce a Quillen equivalence

$$L^2 : s\mathbf{Pre}(\mathscr{C}) \leftrightarrows s\mathbf{Shv}(\mathscr{C}) : i.$$

Proof. The associated sheaf functor L^2 preserves and reflects local weak equivalences. The inclusion functor i preserves injective fibrations and L^2 preserves cofibrations. The associated sheaf map $\eta : X \to L^2X$ is a local weak equivalence, while the counit of the adjunction is an isomorphism. Thus, we have statement 2) if we can prove statement 1).

The axiom **CM1** follows from completeness and cocompleteness for the sheaf category $s\mathbf{Shv}(\mathscr{C})$. The axioms **CM2**, **CM3** and **CM4** follow from the corresponding statements for simplicial presheaves.

A map $p : X \to Y$ is an injective fibration (respectively trivial injective fibration) of $s\mathbf{Shv}(\mathscr{C})$ if and only if it is an injective fibration (respectively trivial injective fibration) of $s\mathbf{Pre}(\mathscr{C})$.

Thus, a simplicial sheaf map p is an injective fibration if and only if it has the right lifting property with respect to all inclusions $A \subset B$ of α-bounded subobjects of $s\mathbf{Shv}(\mathscr{C})$ which are local weak equivalences (recall that the cardinal α is bigger than $|\mathrm{Mor}(\mathscr{C})|$), and it is a trivial injective fibration if and only if it has the right lifting property with respect to all α-bounded cofibrations of $s\mathbf{Shv}(\mathscr{C})$.

The factorization axiom **CM5** is then proved by transfinite small object arguments of size λ where $\lambda > 2^{\alpha}$.

The simplicial model structure is inherited from simplicial presheaves, as is properness.

The injective model structure for simplicial sheaves, which is part 1) of Theorem 5.9, first appeared in a letter of Joyal to Grothendieck [46], while the original demonstration of the injective model structure for simplicial presheaves can be found in [32].

Example 5.10. The category $s\mathbf{Pre}(\mathscr{C})$ of simplicial presheaves is also the category of simplicial sheaves for the “chaotic” Grothendieck topology on $\mathscr{C}$ whose covering sieves are the representable functors $\hom(\ ,U)$, $U \in \mathscr{C}$ (Example 2.9). The injective model structures, for simplicial presheaves or simplicial sheaves, specialize to the injective model structure for diagrams of simplicial sets. The injective model structure for diagrams is the good setting for describing homotopy inverse limits — see [24, VIII.2]. The existence of this model structure is usually attributed to Heller [26], but it is also a consequence of Joyal’s work [46].

5.2 Fibrations and descent

Injective fibrant simplicial presheaves are usually a bit mysterious, but here is a first simple example:

Lemma 5.11. *Suppose that F is a sheaf of sets on $\mathscr{C}$. Then the associated constant simplicial sheaf $K(F,0)$ is injective fibrant.*

The object $K(F,0)$ has n-simplices

$$K(F,0)_n = F,$$

and all simplicial structure maps are the identity on F.

Proof. There is a natural bijection

$$\hom(X, K(F,0)) \cong \hom(\tilde{\pi}_0(X), F)$$

for all simplicial presheaves X and sheaves X. Any local weak equivalence $f : X \to Y$ induces an isomorphims $\tilde{\pi}_0(X) \cong \tilde{\pi}_0(Y)$, and so f induces a bijection

$$f^* : \hom(Y, K(F,0)) \xrightarrow{\cong} \hom(X, K(F,0)).$$

Thus all lifting problems

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & K(F,0) \\ Y \arrow[ur, dotted] & \end{tikzcd}$$

have unique solutions.

Many of the applications of local homotopy theory are based on the sectionwise properties of injective fibrations and injective fibrant objects.

Lemma 5.12. *1) Every injective fibration $p : X \to Y$ is a sectionwise Kan fibration.*

2) Every trivial injective fibration is a sectionwise trivial Kan fibration.

Proof. An injective fibration $p : X \to Y$ has the right lifting property with respect to the trivial cofibrations $L_U\Lambda^n_k \to L_U\Delta^n$. If p is a trivial injective fibration then it has the right lifting property with respect to the cofibrations $L_U\partial\Delta^n \to L_U\Delta^n$.

Corollary 5.13. *Suppose that the map $f : X \to Y$ is a local weak equivalence, and that X and Y are injective fibrant simplicial presheaves. Then f is a sectionwise weak equivalence.*

Proof. The objects X and Y are presheaves of Kan complexes by Lemma 5.12, and are therefore locally fibrant. According to Lemma 5.1, the map f has a factorization

$$\begin{tikzcd} X \arrow[r, "s_*"] \arrow[dr, "f"'] & Z \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

where π is a local fibration and s_* is a section of a local trivial fibration. The map π is a local trivial fibration by Theorem 4.32. It follows from Lemma 5.12 that the maps s_* and π are both sectionwise weak equivalences.

An *injective fibrant model* of a simplicial presheaf X is a local weak equivalence $j : X \to Z$ such that Z is injective fibrant.

A simplicial presheaf X on a site $\mathscr{C}$ is said to satisfy *descent* (or has the *descent property*) if some injective fibrant model $j : X \to Z$ is a sectionwise weak equivalence in the sense that the simplicial set maps $j : X(U) \to Z(U)$ are weak equivalences for all objects U of $\mathscr{C}$.

All injective fibrant objects Z satisfy descent, since any injective fibrant model $Z \to Z'$ is a local weak equivalence between injective fibrant objects, and is therefore a sectionwise weak equivalence by Corollary 5.13.

Corollary 5.14. *Suppose that X satisfies descent, and suppose that $f : X \to W$ is a local weak equivalence such that W is injective fibrant. Then the map f is a sectionwise weak equivalence*

In other words, a simplicial presheaf X satisfies descent if and only if all injective fibrant models $X \to Z$ are sectionwise weak equivalences.

Proof. Suppose that the injective fibrant model $j : X \to Z$ is a sectionwise weak equivalence. We can suppose that the map j is a cofibration by a factorization argument and Corollary 5.13. Let $i : Z \cup_X W \to W'$ be an injective fibrant model for the pushout $Z \cup_X W$. Then by left properness of the injective model structure for simplicial presheaves, all maps in the resulting commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"'] & W \arrow[d] \\ Z \arrow[r] & W' \end{tikzcd}$$

are local weak equivalences, and the objects Z, W and W' are injective fibrant. It follows that the map f is a sectionwise weak equivalence.

This homotopical form of descent is a primary theme in local homotopy theory. All "descent theorems" assert that simplicial presheaves or presheaves of spectra of interest satisfy a descent condition of some form.

Here is a further consequence of the proof of Corollary 5.14:

Corollary 5.15. *Any two injective fibrant models of a simplicial presheaf X are sectionwise weakly equivalent.*

Local fibrations are also very useful in practice.

Lemma 5.16. *Suppose given a pullback diagram*

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ Z \arrow[r] & Y \end{tikzcd} \tag{5.4}$$

in which the map π is a local fibration of simplicial presheaves. Then the diagram is homotopy cartesian for the injective model structure on the category $s\mathbf{Pre}(\mathscr{C})$.

Proof. We use a Boolean localization argument.

The functor $X \mapsto L^2 \operatorname{Ex}^\infty X$ preserves local weak equivalences by Lemma 4.5 and Corollary 4.17, and it is an exercise to show that it preserves local fibrations. We can therefore assume that the pullback diagram is in the category of simplicial sheaves and that all objects are locally fibrant.

Use Theorem 5.9 to find a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\pi"'] & W \arrow[d, "q"] \\ & Y \end{tikzcd}$$

in the category of simplicial sheaves, where q is an injective fibration and j is a cofibration and a local weak equivalence. Then we have to show that the induced map $j_* : Z \times_Y X \to Z \times_Y W$ is a local weak equivalence. The map q is a local fibration by Lemma 5.12.

Boolean localizations preserve and reflect local weak equivalences (Proposition 4.26) and take local local fibrations to sectionwise Kan fibrations (Lemma 4.24). It therefore suffices to assume that all objects are members of a Boolean topos, but in that case the map j_* is a weak equivalence of Kan complexes in each section, again by Lemma 4.24.

Corollary 5.17. *Suppose that $f : X \to Y$ is a local weak equivalence of simplicial presheaves and that A is a simplicial presheaf. Then the map*

$$f \times 1 : X \times A \to Y \times A$$

is a local weak equivalence.

Proof. The functor $X \mapsto \mathrm{Ex}^\infty X$ takes values in locally fibrant simplicial presheaves and preserves products, so it suffices to assume that the objects X, Y and A are locally fibrant. In that case the projection $pr : Y \times A \to Y$ is a local fibration, and the diagram

$$\begin{tikzcd} X \times A \arrow[r, "f \times 1"] \arrow[d, "pr"'] & Y \times A \arrow[d, "pr"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is homotopy cartesian for the injective model structure by Lemma 5.12. It follows that the map $f \times 1$ is a local weak equivalence.

The following result is an immediate consequence.

Corollary 5.18. *Suppose that $i : A \to B$ and $j : C \to D$ are cofibrations of simplicial presheaves. Then the induced cofibration*

$$(B \times C) \cup (A \times D) \to B \times D$$

is a local weak equivalence if either i or j is a local weak equivalence.

If U is an object of the Grothendieck site $\mathscr{C}$, recall that the category $\mathscr{C}/U$ of inherits a topology for which a collection of morphisms $V_i \to V \to U$ is covering for the object $V \to U$ if and only if the morphisms $V_i \to V$ cover the object V of $\mathscr{C}$.

If F is a presheaf on $\mathscr{C}$, write $F|_U$ for the composite

$$(\mathscr{C}/U)^{op} \xrightarrow{q^{op}} (\mathscr{C})^{op} \xrightarrow{F} \mathbf{Set},$$

where $q : \mathscr{C}/U \to \mathscr{C}$ is the canonical functor which takes an object $V \to U$ to V. The presheaf $F|_U$ is the *restriction* of F to $\mathscr{C}/U$, and is a sheaf if F is a sheaf.

If $\phi : U \to U'$ is a morphism of $\mathscr{C}$, then the diagram of functors

$$\begin{tikzcd}
\mathscr{C}/U \arrow[dd, "\phi_*"'] \arrow[dr, "q"] & \\
& \mathscr{C} \\
\mathscr{C}/U' \arrow[ur, "q"'] &
\end{tikzcd}$$

commutes, and so a morphism $E|_{U'} \to F|_{U'}$ restricts to a morphism $E|_U \to F|_U$ by composition with ϕ_*. Thus, there is a presheaf $\mathbf{Hom}(E,F)$ on $\mathscr{C}$ with

$$\mathbf{Hom}(E,F)(U) = \hom(E|_U, F|_U).$$

The presheaf $\mathbf{Hom}(E,F)$ is a sheaf if E and F are sheaves. Observe that a map $E|_U \to F|_U$ can be identified with a presheaf map $E \times U \to F$, where U identified notationally with the representable presheaf $U = \hom(\ ,U)$. We can therefore write

$$\mathbf{Hom}(E,F)(U) = \hom(E \times U, F) \tag{5.5}$$

for all objects U of $\mathscr{C}$.

The description of (5.5) implies that there is an adjunction isomorphism

$$\hom(A, \mathbf{Hom}(E,F)) \cong \hom(A \times E, F) \tag{5.6}$$

for all presheaves A, since every presheaf is a colimit of representables.

If X and Y are simplicial presheaves on $\mathscr{C}$, then the *internal function complex* $\mathbf{Hom}(X,Y)$ is the simplicial presheaf whose U-sections are defined in terms of the function complex on $\mathscr{C}/U$ by the assignment

$$\mathbf{Hom}(X,Y)(U) = \mathbf{hom}(X \times U, Y).$$

There is an exponential law , meaning an isomorphism

$$\hom(A, \mathbf{Hom}(X,Y)) \cong \hom(X \times A, Y)$$

which is natural in simplicial presheaves A, X and Y. This is a consequence of the identifications

$$\hom(U \times \Delta^n, \mathbf{Hom}(X,Y)) \cong \mathbf{Hom}(X,Y)(U)_n = \hom(X \times U \times \Delta^n, Y),$$

and the fact that every simplicial presheaf A is a colimit of objects $U \times \Delta^n$.

The statement of Corollary 5.18 amounts to the existence of an enriched simplicial model structure on the category $s\mathbf{Pre}(\mathscr{C})$. The following is an equivalent formulation:

Corollary 5.19. *Suppose that $p : X \to Y$ is an injective fibration and that $i : A \to B$ is a cofibration of simplicial presheaves. Then the induced map of simplicial presheaves*

$$\mathbf{Hom}(B,X) \to \mathbf{Hom}(A,X) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(A,X)$$

is an injective fibration which is a local weak equivalence if either i or p is a local weak equivalence.

5.3 Geometric and site morphisms

Suppose that $\pi : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Shv}(\mathscr{D})$ is a geometric morphism. Then the inverse image and direct image functors for π induce adjoint functors

$$\pi^* : s\mathbf{Shv}(\mathscr{D}) \leftrightarrows s\mathbf{Shv}(\mathscr{C}) : \pi_*$$

between the respective categories of simplicial sheaves.

Lemma 5.20. *Suppose that $\pi : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Shv}(\mathscr{D})$ is a geometric morphism. Then the inverse image functor*

$$\pi^* : s\mathbf{Shv}(\mathscr{D}) \to s\mathbf{Shv}(\mathscr{C})$$

preserves cofibrations and local weak equivalences.

Proof. The functor π^* is exact, and therefore preserves cofibrations since every monomorphism is an equalizer (Lemma 2.14).

The functor π^* also commutes with the sheaf theoretic Ex^∞-functor, up to natural isomorphism. It therefore suffices to show that π^* preserves local weak equivalences between locally fibrant objects. If $g : X \to Y$ is a local weak equivalence between locally fibrant simplicial sheaves, then g has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "g"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

such that p is a local trivial fibration and the map j is a section of a local trivial fibration, by Lemma 5.1. The inverse image functor π^* preserves local trivial fibrations, by exactness, so that $\pi^*(g)$ is a local weak equivalence of $s\mathbf{Shv}(\mathscr{C})$.

Corollary 5.21. *Suppose that $\pi : \mathbf{Shv}(\mathscr{C}) \to \mathbf{Shv}(\mathscr{D})$ is a geometric morphism. Then the adjoint functors*

$$\pi^* : s\mathbf{Shv}(\mathscr{D}) \leftrightarrows s\mathbf{Shv}(\mathscr{C}) : \pi_*$$

form a Quillen adjunction for the injective model structures on the respective categories of simplicial sheaves. In particular if X is an injective fibrant simplicial sheaf on $\mathscr{C}$, then its inverse image $\pi_* X$ is injective fibrant.

Lemma 5.22. *Suppose that the functor $f : \mathscr{C} \to \mathscr{D}$ is a site morphism. Then the inverse image functor*

$$f^p : s\mathbf{Pre}(\mathscr{C}) \to s\mathbf{Pre}(\mathscr{D})$$

preserves cofibrations and local weak equivalences.

Proof. The proof is similar to that of Lemma 5.20. Every monomorphism of $s\mathbf{Pre}(\mathscr{C})$ is an equalizer and f^p preserves equalizers, so that f^p preserves monomorphisms. The functor f^p commutes with Kan's Ex^∞ functor up to natural isomorphism, by exactness, and preserves local epimorphisms since the functor f^* preserves epimorphisms (use Lemma 2.16). It follows that f^p preserves local trivial fibrations. Finish by using the factorization of Lemma 5.1.

Corollary 5.23. *Suppose that the functor $f : \mathscr{C} \to \mathscr{D}$ is a site morphism. Then the adjoint functors*

$$f^p : s\mathbf{Pre}(\mathscr{C}) \leftrightarrows s\mathbf{Pre}(\mathscr{D}) : f_*$$

form a Quillen adjunction for the respective injective model structures. In particular, the functor f_* preserves injective fibrant objects.

The assertion that the direct image functor f_* preserves injective fibrant objects first appeared in [33], with essentially the same proof.

The *forgetful functor* $q : \mathscr{C}/U \to \mathscr{C}$ is defined on objects by

$$q(V \xrightarrow{\phi} U) = V.$$

This functor is continuous for the topology on $\mathscr{C}/U$ which is inherited from the site $\mathscr{C}$, but it is not necessarily a site morphism. We nevertheless have the following useful result:

Lemma 5.24. *Suppose that $\mathscr{C}$ is a Grothendieck site and that U is an object of $\mathscr{C}$. Then the functor*

$$q^p : s\mathbf{Pre}(\mathscr{C}/U) \to s\mathbf{Pre}(\mathscr{C})$$

preserves cofibrations and local weak equivalences.

Proof. The functor q^p is defined, for a simplicial presheaf X on $\mathscr{C}/U$, by

$$q^p(X)(V) = \bigsqcup_{\phi : V \to U} X(\phi)$$

for $V \in \mathscr{C}$. This functor plainly preserves cofibrations.

Suppose that $p : X \to Y$ is a locally trivial fibration on $\mathscr{C}/U$ and that there is a commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & q^p X(V) \arrow[d, "p_*"] \\
\Delta^n \arrow[r] & q^p Y(V)
\end{tikzcd}$$

The Δ^n is connected for all $n \geq 0$, so that there is a factorization of this diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X(\phi) \arrow[r, "in_\phi"] \arrow[d, "p"] & \sqcup_{\phi:V\to U} X(\phi) \arrow[d, "p_*"] \\
\Delta^n \arrow[r] & Y(\phi) \arrow[r, "in_\phi"'] & \sqcup_{\phi:V\to U} Y(\phi)
\end{tikzcd}$$

for some map $\phi : V \to U$, where in_ϕ is the inclusion of the summand corresponding to the map ϕ. There is a covering

$$\begin{tikzcd}
V_i \arrow[r] \arrow[dr, "\phi_i"'] & V \arrow[d, "\phi"] \\
& U
\end{tikzcd}$$

of ϕ such that the liftings exist in the diagrams

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X(\phi) \arrow[r] & X(\phi_i) \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[urr, dotted] & Y(\phi) \arrow[r] & Y(\phi_i)
\end{tikzcd}$$

It follows that the liftings exist in the diagrams

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & q^p X(V) \arrow[r] & q^p X(V_i) \arrow[d, "p_*"] \\
\Delta^n \arrow[r] \arrow[urr, dotted] & q^p Y(V) \arrow[r] & q^p Y(V_i)
\end{tikzcd}$$

after refinement along the covering $V_i \to V$.

The functor q^p therefore preserves local trivial fibrations. It also commutes up to isomorphism with the Ex^∞ functor. It follows from Lemma 5.1 that q^p preserves local weak equivalences.

Corollary 5.25. *Suppose that $\mathscr{C}$ is a Grothendieck site, U is an object of $\mathscr{C}$ and that $q : \mathscr{C}/U \to \mathscr{C}$ is the forgetful functor. Then the adjoint functors*

$$q^p : s\mathbf{Pre}(\mathscr{C}/U) \leftrightarrows s\mathbf{Pre}(\mathscr{C}) : q_*$$

define a Quillen adjunction for the respective injective model structures. In particular, the restriction functor

$$X \mapsto q_*(X) = X|_U$$

preserves injective fibrant objects.

The presheaf-level restriction functor

$$q_* : \mathbf{Pre}(\mathscr{C}) \to \mathbf{Pre}(\mathscr{C}/U)$$

is exact (preserves limits and colimits) and preserves local epimorphisms. It follows that the functor

$$q_* : s\mathbf{Pre}(\mathscr{C}) \to s\mathbf{Pre}(\mathscr{C}/U)$$

commutes with the Ex^∞ functor up to natural isomorphism and preserves local fibrations and local trivial fibrations. In particular, the restriction functor q_* preserves local weak equivalences, by Lemma 5.1. We also have the following:

Lemma 5.26. *A map $f : X \to Y$ is a local fibration (respectively local trivial fibration, respectively local weak equivalence) if and only if the restrictions $f|_U : X|_U \to Y|_U$ are local fibrations (respectively local trivial fibrations, respectively local weak equivalences) for all objects $U \in \mathscr{C}$.*

Proof. A map $F \to G$ of presheaves on $\mathscr{C}$ is a local epimorphism if and only if the restrictions $F|_U \to G|_U$ are local epimorphisms on $\mathscr{C}/U$ for all $U \in \mathscr{C}$. The claims about local fibrations and local trivial fibrations follow immediately. One proves the claim about local weak equivalences with another appeal to Lemma 5.1.

Example 5.27. The ideas of this section occur frequently in examples.

Suppose that $f : T \to S$ is a morphism of schemes which is locally of finite type. Then f occurs as an object of the big étale site $(Sch|_S)_{et}$. Pullback along the scheme homomorphism f determines a site morphism

$$f : (Sch|_S)_{et} \to (Sch|_T)_{et}.$$

One can identify the site $(Sch|_T)_{et}$ with the slice category $(Sch|_S)_{et}/f$, and the presheaf-level inverse image functor

$$f^p : \mathbf{Pre}((Sch|_S)_{et}) \to \mathbf{Pre}((Sch|_T)_{et})$$

(ie. the left adjoint of composition with the pullback functor) is isomorphic to the restriction functor which is induced by composition with f. It follows from Corollary 5.25 that the functor

$$f^p : s\mathbf{Pre}((Sch|_S)_{et}) \to s\mathbf{Pre}((Sch|_T)_{et})$$

preserves injective fibrations. This functor also preserves local weak equivalences since it is an inverse image functor for a site morphism.

The inclusion $i : et|_T \subset (Sch|_T)_{et}$ of the étale site in the big étale site is a site morphism for each S-scheme T. Restriction to $et|_T$ is exact and preserves local epimorphisms for presheaves on $(Sch|_T)_{et}$, and it therefore preserves local weak equivalences. This restriction functor also preserves injective fibrations, by Corollary 5.23.

It follows that composite restriction functors

$$s\mathbf{Pre}((Sch|_S)_{et} \xrightarrow{f^p} s\mathbf{Pre}((Sch|_T)_{et} \xrightarrow{i_*} s\mathbf{Pre}((et|_T))$$

preserve local weak equivalences and injective fibrations for all S-schemes $f : T \to S$.

These functors are exact and commute with the Ex^∞ construction. Taken together, these functors reflect local epimorphisms. Thus a simplicial presheaf map $X \to Y$ on the big étale site for S is a local weak equivalence if and only if the induced map $i_* f^p(X) \to i_* f^p(Y)$ is a local weak equivalence on the ordinary étale site $et|_T$ for each S-scheme $f : T \to S$.

The foregoing is only a paradigm. Similar arguments and results are available for the flat, Zariski and Nisnevich topologies (for example), and for variations of the big site such as the smooth site. These results are very useful for cohomology calculations.

We close this section with a general result (Proposition 5.28) about simplicial objects S in a site $\mathscr{C}$; this result is effectively a non-abelian version of a cohomology isomorphism that will occur in Chapter 7. In colloquial terms, this result asserts that cohomological invariants for such an object S can be computed either in simplicial presheaves on $\mathscr{C}$, or in simplicial presheaves on a site $\mathscr{C}/S$ which is fibred over S. The latter is the usual setting for the classical approach to the cohomology of simplicial schemes [16]. Proposition 5.28 is proved by using a descent argument.

Suppose that S is a simplicial object in the site $\mathscr{C}$. The site $\mathscr{C}/S$ *fibred over* S has for objects all morphisms $U \to S_n$, and for morphisms all commutative diagrams

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d] & V \arrow[d] \\ S_n \arrow[r, "\theta^*"'] & S_m \end{tikzcd} \tag{5.7}$$

where θ^* is a simplicial structure map. The covering families of the site $\mathscr{C}$ are the families

$$\begin{tikzcd} U_i \arrow[r, "\phi_i"] \arrow[d] & U \arrow[d] \\ S_n \arrow[r, "1"'] & S_n \end{tikzcd}$$

where the family $U_i \to U$ is covering for U in $\mathscr{C}$.

There is a simplicial object 1_S in $\mathscr{C}/Y$, with n-simplices given by the identity $1 : S_n \to S_n$, and with the diagrams

$$\begin{tikzcd} S_n \arrow[r, "\theta^*"] \arrow[d, "1"'] & S_m \arrow[d, "1"] \\ S_n \arrow[r, "\theta^*"'] & S_m \end{tikzcd}$$

as structure maps. This simplicial object represents a simplicial presheaf on $\mathscr{C}/S$, which will also be denoted by 1_S.

There is a functor $q : \mathscr{C}/S \to \mathscr{C}$ which takes the morphism (5.7) to the morphism $\phi : U \to V$ of $\mathscr{C}$. Composition with q defines a restriction functor

$$q_* : s\mathbf{Pre}(\mathscr{C}) \to s\mathbf{Pre}(\mathscr{C}/S),$$

and we write

$$X|_S = q_*(X)$$

for simplicial presheaves X on $\mathscr{C}$.

There are obvious inclusions $j_n : \mathscr{C}/S_n \to \mathscr{C}/S$ which induce restriction functors

$$j_{n*} : s\mathbf{Pre}(\mathscr{C}/S) \to s\mathbf{Pre}(\mathscr{C}/S_n)$$

by precomposition. The composite

$$\mathscr{C}/S_n \xrightarrow{j_n} \mathscr{C}/S \xrightarrow{q} \mathscr{C}$$

is an instance of the forgetful functor $\mathscr{C}/S_n \to \mathscr{C}$ (see Lemma 5.24). We shall denote this composite functor by q_n.

Precomposition with j_n defines a restriction functor

$$j_{n*} : s\mathbf{Pre}(\mathscr{C}/S) \to s\mathbf{Pre}(\mathscr{C}/S_n).$$

The restriction functor j_{n*} has a left adjoint

$$j_n^p : s\mathbf{Pre}(\mathscr{C}/S_n) \to s\mathbf{Pre}(\mathscr{C}/S)$$

which is defined for a simplicial presheaf X by

$$j_n^p(X)(U \xrightarrow{\phi} S_m) = \bigsqcup_{\mathbf{n} \xrightarrow{\theta} \mathbf{m}} X(U \xrightarrow{\phi} S_m \xrightarrow{\theta^*} S_n).$$

This functor j_n^p preserves cofibrations. The functors j_n^p and j_{n*} both commute with the Ex^∞ functor and preserve local trivial fibrations, and therefore both preserve local weak equivalences. It follows in particular that j_{n*} preserves injective fibrant models.

We also conclude that a map $f : X \to Y$ of simplicial presheaves on $\mathscr{C}/S$ is a local weak equivalence if and only if the restrictions $j_{n*}f : j_{n*}X \to j_{n*}Y$ are local weak equivalences on $\mathscr{C}/S_n$ for all n.

Proposition 5.28. *Suppose that S is a simplicial object in a site $\mathscr{C}$ and that Z is an injective fibrant simplicial presheaf on $\mathscr{C}$. Choose an injective fibrant model $j : Z|_S \to W$ on $\mathscr{C}/S$. Then there is a weak equivalence*

$$\mathbf{hom}(S,Z) \cong \mathbf{hom}(*,W).$$

This weak equivalence is natural in the map $j : Z|_S \to W$.

Proof. There is an isomorphism

$$1_S(U \to S_m) \cong \Delta^m$$

of simplicial sets. It follows that the map $1_S \to *$ is a sectionwise weak equivalence on $\mathscr{C}/S$.

The restricted object $Z|_S$ satisfies descent. In effect, the restricted map

$$j_* : j_{n*}(Z|_S) \to j_{n*}(W)$$

is a local weak equivalence of simplicial presheaves on $\mathscr{C}/S_n$ for all $n \geq 0$. The restriction $j_{n*}(Z|_S) = q_{n*}(Z)$ is injective fibrant on $\mathscr{C}/S_n$ for all $n \geq 0$ by Lemma 5.24 and the discussion above. Local weak equivalences of injective fibrant objects are sectionwise weak equivalences, and it follows that the maps

$$Z|_S(\phi) \to W(\phi)$$

are weak equivalences of simplicial sets for all objects $\phi : U \to S_n$ of $\mathscr{C}/S$.

It follows that the maps

$$Z|_S(1_{S_n}) \to W(1_{S_n})$$

are weak equivlences for all $n \geq 0$. There is an isomorphism

$$Z|_S(1_{S_n}) \cong Z(S_n)$$

of cosimplicial spaces. It follows from Lemma 5.29 below that there is a weak equivalence

$$\underleftarrow{\mathrm{holim}}_n\, Z|_S(1_{S_n}) \simeq \mathbf{hom}(S,Z).$$

There are also weak equivalences

$$\underleftarrow{\mathrm{holim}}_n\, Z|_S(1_{S_n}) \xrightarrow{\simeq} \underleftarrow{\mathrm{holim}}_n\, W(1_{S_n}) \simeq \mathbf{hom}(1_S,W) \xleftarrow{\simeq} \mathbf{hom}(*,W)$$

since W is injective fibrant on $\mathscr{C}/S$ and the map $1_S \to *$ is a local weak equivalence.

Lemma 5.29. *Suppose that the simplicial presheaf S is represented by a simplicial object in the site $\mathscr{C}$, and suppose that Z is an injective fibrant simplicial presheaf. Then there is a weak equivalence*

$$\mathbf{hom}(S,Z) \simeq \underleftarrow{\mathrm{holim}}_n\, Z(S_n).$$

Proof. Let $Z(S)$ be the cosimplicial space with

$$Z(S)^n = Z(S_n)$$

for $n \geq 0$.

There is a natural bijection

$$\hom(A, Z(S)) \cong \hom(A \otimes S, Z)$$

relating morphisms of cosimplicial spaces to morphisms of simplicial presheaves.

Here, $A \otimes S$ is a coend in the sense that it is described by the coequalizer

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} A^m \times S_n \rightrightarrows \bigsqcup_{\mathbf{n}} A^n \times S_n \to A \otimes S$$

in simplicial presheaves. Observe that the simplicial presheaf S also is a coend, in that there is a coequalizer

$$\bigsqcup_{\theta:\mathbf{m}\to\mathbf{n}} \Delta^m \times S_n \rightrightarrows \bigsqcup_{\mathbf{n}} \Delta^n \times S_n \to S,$$

so that there is an isomorphism $S \cong \Delta \otimes S$.

A cosimplicial space map $\Delta \times \Delta^n \to Z(S)$ therefore corresponds uniquely to a simplicial presheaf map

$$S \times \Delta^n \cong (\Delta \otimes S) \times \Delta^n \to Z,$$

and it follows that there is a natural isomorphism

$$\mathrm{Tot}\, Z(S) \cong \mathbf{hom}(S,Z).$$

The degenerate part DS_n of the presheaf S_n is a subobject of S_n, and is defined by a coequalizer

$$\bigsqcup_{i<j} S_{n-2} \rightrightarrows \bigsqcup_{i} S_{n-1} \xrightarrow{s} DS_n,$$

where the map s is induced by the degeneracy $s_i : S_{n-1} \to S_n$ on the summand corresponding to i. The cofibration $DS_n \subset S_n$ induces a Kan fibration

$$Z(S)^n \cong \mathbf{hom}(S_n, Z) \to \mathbf{hom}(DS_n, Z) = M^{n-1}Z(S)$$

since Z is injective fibrant. The cosimplicial space $Z(S)$ is therefore Bousfield-Kan fibrant [8, X.4.6], and so the canonical map

$$\mathrm{Tot}\ Z(S) \to \underleftarrow{\mathrm{holim}}_n\, Z(S_n)$$

is a weak equivalence of simplicial sets [8, XI.4.4].

Remark 5.30. The homotopy inverse limit for a cosimplicial space X can be defined by

$$\underleftarrow{\mathrm{holim}}_n\, X = \varprojlim_n Z,$$

where $j : X \to Z$ is an injective fibrant model for X in the category of cosimplicial spaces. Every injective fibrant cosimplicial space is Bousfield-Kan fibrant. Thus, if X is also Bousfield-Kan fibrant then the map j induces a weak equivalence

$$\mathrm{Tot}\ X \xrightarrow{\simeq} \mathrm{Tot}\ Z = \mathbf{hom}(\Delta, Z).$$

The map $\Delta \to *$ is a weak equivalence of cosimplicial spaces and Z is injective fibrant, and so there is a weak equivalence

$$\varprojlim_n Z = \mathbf{hom}(*, Z) \xrightarrow{\simeq} \mathbf{hom}(\Delta, Z).$$

5.4 Cocycles

Let $\mathscr{M}$ be a closed model category such that

1) $\mathscr{M}$ is right proper in the sense that weak equivalences pull back to weak equivalences along fibrations, and
2) the class of weak equivalences is closed under finite products: if $f : X \to Y$ is a weak equivalence, so is any map $f \times 1 : X \times Z \to Y \times Z$

Examples include all of the model structures on simplicial presheaves and simplicial sheaves that we've seen so far, where the weak equivalences are local weak equivalences. In effect, these model structures are proper (Theorem 5.8, Theorem 5.9), and weak equivalences are closed under finite products by Lemma 5.17.

Suppose that X, Y are objects of $\mathscr{M}$, and write $h(X,Y)$ for the category whose objects are all pairs of maps (f, g)

$$X \xleftarrow[\simeq]{f} Z \xrightarrow{g} Y$$

where f is a weak equivalence. A morphism

$$\gamma : (f, g) \to (f', g')$$

of $h(X,Y)$ is a map $\gamma : Z \to Z'$ which makes the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"', "\simeq"] \arrow[dr, "g"] \arrow[d, "\gamma"] & \\ X & & Y \\ & Z' \arrow[ul, "f'", "\simeq"'] \arrow[ur, "g'"'] & \end{tikzcd}$$

commute. The category $h(X,Y)$ is the *category of cocycles*, or *cocycle category*, from X to Y. The objects of $h(X,Y)$ are called *cocycles*.

Example 5.31. Suppose that a presheaf map $U \to *$ is a local epimorphism, and recall from Example 4.18 that the canonical simplicial presheaf map

$$\check{C}(U) = BC(U) \to *$$

is a local weak equivalence (in fact, it's a local trivial fibration). The object $\check{C}(U)$ is the *Čech resolution* associated to the covering $U \to *$.

Given a covering $U \to *$ and a (pre)sheaf of groups G, a *normalized cocycle* on U with values in G is, precisely, either a morphism $C(U) \to G$ of presheaves of groupoids or a simplicial presheaf map $BC(U) \to BG$. Such a map defines a cocycle

$$* \xleftarrow{\simeq} BC(U) \to BG$$

in the sense described above. Normalized cocycles are the original examples of the cocycles described here.

Write $\pi_0 h(X,Y)$ for the path components of the category $h(X,Y)$. There is a function

$$\phi : \pi_0 h(X,Y) \to [X,Y]$$

which is defined by the assignment $(f,g) \mapsto g \cdot f^{-1}$.

Lemma 5.32. *Suppose that $\gamma : X \to X'$ and $\omega : Y \to Y'$ are weak equivalences. Then the function*

$$(\gamma,\omega)_* : \pi_0 h(X,Y) \to \pi_0 h(X',Y')$$

is a bijection.

Proof. An object (f,g) of $h(X',Y')$ is a map $(f,g) : Z \to X' \times Y'$ such that f is a weak equivalence. There is a factorization

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "(f,g)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ & X' \times Y' \end{tikzcd}$$

such that j is a trivial cofibration and $(p_{X'}, p_{Y'})$ is a fibration. The map $p_{X'}$ is a weak equivalence. Form the pullback

$$\begin{tikzcd} W_* \arrow[r, "(\gamma\times\omega)_*"] \arrow[d, "(p_X^*,p_Y^*)"'] & W \arrow[d, "(p_{X'},p_{Y'})"] \\ X \times Y \arrow[r, "\gamma\times\omega"'] & X' \times Y' \end{tikzcd}$$

Then the map (p_X^*, p_Y^*) is a fibration and $(\gamma \times \omega)_*$ is a local weak equivalence since $\gamma \times \omega$ is a weak equivalence, by right properness. The map p_X^* is also a weak equivalence.

The assignment $(f,g) \mapsto (p_X^*, p_Y^*)$ defines a function

$$\pi_0 h(X',Y') \to \pi_0 h(X,Y)$$

which is inverse to $(\gamma, \omega)_*$.

Lemma 5.33. *Suppose that Y is fibrant and X is cofibrant. Then the canonical function*

$$\phi : \pi_0 h(X,Y) \to [X,Y]$$

is a bijection.

Proof. The function $\pi(X,Y) \to [X,Y]$ relating homotopy classes of maps $X \to Y$ to morphisms in the homotopy category is a bijection since X is cofibrant and Y is fibrant.

If $f,g : X \to Y$ are homotopic, there is a diagram

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[d, "d_0"'] \arrow[dr, "f"] & \\ X & X \otimes I \arrow[l, "s"'] \arrow[r, "h"] & Y \\ & X \arrow[ul, "1"] \arrow[u, "d_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

where h is the homotopy. Thus, sending $f : X \to Y$ to the class of $(1_X, f)$ defines a function

$$\psi : \pi(X,Y) \to \pi_0 h(X,Y)$$

and there is a diagram

$$\begin{tikzcd} \pi(X,Y) \arrow[r, "\psi"] \arrow[dr, "\cong"'] & \pi_0 h(X,Y) \arrow[d, "\phi"] \\ & {[X,Y]} \end{tikzcd}$$

It suffices to show that ψ is surjective, or that any object $X \xleftarrow{f} Z \xrightarrow{g} Y$ is in the path component of some a pair $X \xleftarrow{1} X \xrightarrow{k} Y$ for some map k.

The weak equivalence f has a factorization

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "f"'] & V \arrow[d, "p"] \\ & X \end{tikzcd}$$

where j is a trivial cofibration and p is a trivial fibration. The object Y is fibrant, so the dotted arrow θ exists in the diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, "j"] & \\ X & V \arrow[l, "p"] \arrow[r, dotted, "\theta"'] & Y \end{tikzcd}$$

Since X is cofibrant, the trivial fibration p has a section σ, and so there is a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "1"'] \arrow[dr, "\theta\sigma"] \arrow[d, "\sigma"] & \\ X & V \arrow[l, "p"] \arrow[r, dotted, "\theta"'] & Y \end{tikzcd}$$

Then the composite $\theta\sigma$ is the required map k.

Theorem 5.34. *Suppose that the model category $\mathscr{M}$ is right proper and has weak equivalences closed under finite products. Suppose that X,Y are objects of $\mathscr{M}$. Then the canonical function*

$$\phi : \pi_0 h(X,Y) \to [X,Y]$$

is a bijection.

Proof. There are weak equivalences $\pi : X' \to X$ and $j : Y \to Y'$ such that X' and Y' are cofibrant and fibrant, respectively, and there is a commutative diagram

$$\begin{tikzcd} \pi_0 h(X,Y) \arrow[r, "\phi"] \arrow[d, "(1,j)_*"', "\cong"] & {[X,Y]} \arrow[d, "j_*", "\cong"'] \\ \pi_0 h(X,Y') \arrow[r, "\phi"] & {[X,Y']} \arrow[d, "\pi^*", "\cong"'] \\ \pi_0 h(X',Y') \arrow[u, "(\pi,1)_*", "\cong"'] \arrow[r, "\cong", "\phi"'] & {[X',Y']} \end{tikzcd}$$

The functions $(1,j)_*$ and $(\pi,1)_*$ are bijections by Lemma 5.32, and the bottom map ϕ is a bijection by Lemma 5.33.

Remark 5.35. Cocycle categories have appeared before, in the context of Dwyer-Kan hammock localizations [14], [13]. One of the main results of the theory, which holds for arbitrary model categories $\mathscr{M}$, says roughly that the nerve $Bh(X,Y)$ is a model for the function space of maps from X to Y if Y is fibrant. This result implies Theorem 5.34 if the target object Y is fibrant. On the other hand, we will see below that the most powerful applications of Theorem 5.34 involve target objects Y which are not fibrant in general.

The statement of Theorem 5.34 must be interpreted with some care because the cocycle category $h(X,Y)$ may not be small. The Theorem says that two cocycles are in the same path component in the sense that they are connected by a finite string of morphisms of $h(X,Y)$ if and only if they represent the same morphism in the homotopy category, and that every morphism in the homotopy category can be represented by a cocycle. Similar care is required for the interpretation of Lemma 5.32 and Lemma 5.33 in general.

For simplicial presheaves (and simplicial sheaves), we have the following:

Proposition 5.36. *Suppose that a simplicial presheaf X is α-bounded, where α is an infinite cardinal such that $\alpha > |\mathrm{Mor}(\mathscr{C})|$, where $\mathscr{C}$ is the underlying small site. Let $h(X,Y)_\alpha$ be the full subcategory of $h(X,Y)$ on those cocycles*

$$X \xleftarrow{\simeq} U \to Y$$

such that U is α-bounded . Then the induced function

$$\pi_0 h(X,Y)_\alpha \to \pi_0 h(X,Y)$$

is a bijection.

Observe that the category $h(X,Y)_\alpha$ in the statement of Proposition 5.36 is small. The proof of this result uses the following technical lemmas:

Lemma 5.37. *Suppose that $i : A \to B$ is a cofibration such that A is α-bounded. Then there is an α-bounded subobject $C \subset B$ with $A \subset B$ such that all presheaf maps $\pi_* C \to \pi_* B$ are monomorphisms.*

Proof. The simplicial presheaf $\mathrm{Ex}^\infty B$ is a filtered colimit of simplicial presheaves $\mathrm{Ex}^\infty D$, where D varies through the α-bounded subcomplexes of B. Any commutative diagram

$$\begin{tikzcd}
\partial\Delta^{n+1} \arrow[r, "{(\gamma,*,\dots,*)}"] \arrow[d] & \mathrm{Ex}^\infty A(U) \arrow[d] \\
\Delta^{n+1} \arrow[r] & \mathrm{Ex}^\infty B(U)
\end{tikzcd}$$

therefore factors through $\mathrm{Ex}^\infty D(U)$, where D is an α-bounded subobject of B. Such lifting problems are indexed on simplices γ of A which represent homotopy group elements, so there is an α-bounded subcomplex $A_1 \subset B$ such that all diagrams as above factor through A_1. Repeat this process inductively to produce a string of inclusions

$$A = A_0 \subset A_1 \subset A_2 \subset \dots$$

of α bounded subcomplexes of B. Then the subcomplex $C = \bigcup_i A_i$ is α-bounded, and the presheaf maps $\pi_* C(U) \to \pi_* B(U)$ are monomorphisms.

Lemma 5.38. *Suppose given a diagram*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "h"'] & Y \arrow[d, "f"] \\ & Z \end{tikzcd}$$

of simplicial presheaf maps such that h is a local weak equivalence and the induced maps $f_* : \tilde{\pi}_n Y \to \tilde{\pi}_n Z$ *are monomorphisms of sheaves for* $n \geq 0$*. Then the map* f *is a local weak equivalence.*

Proof. The analogous claim for morphisms of Kan complexes is true. In that case, we can suppose that f is a Kan fibration, and then we show that f has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$.

In general, we can assume that X, Y and Z are locally fibrant simplicial sheaves and that f is a local fibration. Take a Boolean localization $p : \mathbf{Shv}(\mathscr{B}) \to \mathbf{Shv}(\mathscr{C})$, and observe that the induced diagram

$$\begin{tikzcd} p^*X \arrow[r, "g_*"] \arrow[dr, "h_*"'] & p^*Y \arrow[d, "f_*"] \\ & p^*Z \end{tikzcd}$$

of simplicial sheaf maps on $\mathscr{B}$ is a map of diagrams of Kan complexes which satisfies the conditions of the Lemma in all sections.

Proof (Proof of Proposition 5.36). Suppose that

$$X \xleftarrow[\simeq]{g} U \to Y$$

is a cocycle, and that X is α-bounded. The map $g : U \to Y$ has a factorization

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[dr, "g"'] & Z \arrow[d, "p"] \\ & X \end{tikzcd}$$

where i is a trivial cofibration and p is a trivial injective fibration. The map p has a section $\sigma : X \to Z$ which is a trivial cofibration. There is an α-bounded subobject X_1 of Z which contains X such that the induced map $X_1 \cap U \to X_1$ is a weak equivalence. There is an α-bounded subobject X_1' of Z which contains X_1 such that the cofibration $X_1' \to Z$ is a weak equivalence, by Lemma 5.37 and a Boolean localization argument — see the proof of Proposition 4.26.

Repeat these constructions inductively, to form a sequence of cofibrations

$$X \subset X_1 \subset X_1' \subset X_2 \subset X_2' \subset \dots$$

between α-bounded subobjects of X, and let A be the union of these subobjects. Then, A is α-bounded, and by cofinality the map $A \subset Z$ is a weak equivalence, as is the map $A \cap U \to A$.

It follows that there is a commutative diagram

$$\begin{tikzcd} A \arrow[d, "\simeq"'] & A \cap U \arrow[l, "\simeq"] \arrow[d, "\simeq"] & \\ X & U \arrow[l, "\simeq", "g"'] \arrow[r] & Y \end{tikzcd}$$

where $A \cap U$ is α-bounded, so that the map $\pi_0 h(X,Y)_\alpha \to \pi_0 h(X,Y)$ is surjective.

Suppose given a diagram

$$\begin{tikzcd} & A \arrow[d, "\simeq"', "i"] & \\ & U \arrow[dl, "g"', "\simeq"] \arrow[dr, "f"] \arrow[dd, "\theta"] & \\ X & & Y \\ & U' \arrow[ul, "g'", "\simeq"'] \arrow[ur, "f'"'] & \\ & A' \arrow[u, "\simeq", "i'"'] & \end{tikzcd}$$

where the maps i and i' are trivial cofibrations with A and A' α-bounded. Then the subobject $\theta(A) \subset U'$ is α-bounded, as is the union $A' \cup \theta(A)$. By Lemma 5.37 (and Boolean localization), there is an α-bounded subobject B of U' with $A' \cup \theta(A) \subset B$ and such that $B \subset U'$ is a weak equivalence. The cocycles (gi, fi) and $(g'i', f'i')$ are therefore in the same path component of $h(X,Y)_\alpha$. It also follows that the map $\pi_0 h(X,Y)_\alpha \to \pi_0 h(X,Y)$ is injective.

We shall also need the following result:

Lemma 5.39. *Suppose that the simplicial presheaf X is α-bounded, where α is an infinite cardinal such that $\alpha > |\mathrm{Mor}(\mathscr{C})|$. Suppose that β is a cardinal such that $\beta > \alpha$. Then the inclusion functor $j : h(X,Y)_\alpha \subset h(X,Y)_\beta$ induces a weak equivalence*

$$j_* : Bh(X,Y)_\alpha \simeq Bh(X,Y)_\beta .$$

Proof. Suppose that

$$X \xleftarrow[\simeq]{f} V \xrightarrow{g} Y$$

is a cocycle such that V is β-bounded. We show that the slice category $j/(f,g)$ has a contractible nerve. The Lemma then follows from Quillen's Theorem B (or Theorem A) [24, IV.5.6], [57].

The category $j/(f,g)$ is isomorphic to the category we_α/V whose objects are the local weak equivalences $\theta : U \to V$ with U α-bounded, and whose morphisms $\theta \to \theta'$ are commutative diagrams

$$\begin{tikzcd} U \arrow[rr] \arrow[dr, "\theta"'] & & U' \arrow[dl, "\theta'"] \\ & V & \end{tikzcd}$$

of simplicial presheaf morphisms. Write cof_α/V for the full subcategory of we_α/V whose objects are the cofibrations, and let $i : cof_\alpha/V \subset we_\alpha/V$ be the inclusion functor.

The slice category θ/i is non-empty. In effect, the image $\theta(U)$ of the weak equivalence θ is an α-bounded subobject of V, $\theta(U)$ is contained in an α-bounded subobject A of V such that $\pi_*(A) \to \pi_*(V)$ is a monomorphism of presheaves by Lemma 5.37, and then the inclusion $A \subset V$ is a weak equivalence by Lemma 5.38. The category θ/i is also filtered, again by Lemmas 5.37 and 5.38.

This is true for all $\theta : U \to V$ in the category we_α/V, so that the induced map

$$i_* : B(cof_\alpha/V) \subset B(we_\alpha/V)$$

is a weak equivalence.

Finally, the category cof_α/V is non-empty filtered by Lemmas 5.37 and 5.38, and it follows that the simplicial set $B(we_\alpha/V)$ is contractible.

Corollary 5.40. *Suppose that $f : X \to X'$ is a local weak equivalence of α-bounded simplicial presheaves, where α is an infinite cardinal such that $\alpha > |\mathrm{Mor}(\mathscr{C})|$. Suppose that $g : Y \to Y'$ is a local weak equivalence. Then the induced simplicial set map*

$$(f,g)_* : Bh(X,Y)_\alpha \to Bh(X',Y')_\alpha$$

is a weak equivalence.

Proof. Following the proof of Lemma 5.32, suppose that $(f,g) : Z \to X' \times Y'$ is an α-bounded cocycle, and take the functorial factorization

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "{(f,g)}"'] & W \arrow[d, "{p=(p_{X'},p_{Y'})}"] \\ & X' \times Y' \end{tikzcd}$$

such that j is a trivial cofibration and p is an injective fibration. Form the pullback diagram

$$\begin{tikzcd} W_* \arrow[r, "(\gamma\times\omega)_*"] \arrow[d, "{(p_X^*,p_Y^*)}"'] & W \arrow[d, "{(p_{X'},p_{Y'})}"] \\ X \times Y \arrow[r, "\gamma\times\omega"'] & X' \times Y' \end{tikzcd}$$

as before. Then there is a cardinal $\beta > \alpha$ such that all objects in this diagram are β-bounded, and of course the map (p_X^*, p_Y^*) is a cocycle since the map $(\gamma \times \omega)_*$ is a local weak equivalence. It follows that there is a homotopy commutative diagram

$$\begin{tikzcd} Bh(X,Y)_\alpha \arrow[r, "(\gamma,\omega)_*"] \arrow[d, "\simeq"'] & B(X',Y')_\alpha \arrow[d, "\simeq"] \arrow[dl] \\ Bh(X,Y)_\beta \arrow[r, "(\gamma,\omega)_*"'] & B(X',Y')_\beta \end{tikzcd}$$

of simplicial set maps in which the vertical maps are weak equivalences by Lemma 5.39. The statement of the Corollary follows.

5.5 The Verdier hypercovering theorem

The discussion that follows will be confined to simplicial presheaves. It has an exact analog for simplicial sheaves.

Recall that a hypercover $p : Z \to X$ is a locally trivial fibration. This means, equivalently (Theorem 4.32), that p is a local fibration and a local weak equivalence, or that p has the local right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.

The objects of the category $Triv/X$ are the simplicial homotopy classes of maps $[p] : Z \to X$ which are represented by hypercovers $p : Z \to X$. The morphisms of this category are commutative triangles of simplicial homotopy classes of maps in the obvious sense.

To be completely explicit, suppose that $f, g : Z \to X$ are simplicial presheaf morphisms, and recall that a simplicial homotopy from f to g is a commutative diagram of simplicial presheaf maps

$$\begin{tikzcd} Z \arrow[d, "d^1"'] \arrow[dr, "f"] & \\ Z \times \Delta^1 \arrow[r] & X \\ Z \arrow[u, "d^0"] \arrow[ur, "g"'] & \end{tikzcd}$$

We say that f and g are simplicially homotopic and write $f \sim g$. The set $\pi(Z,X)$ is the effect of collapsing the morphism set $\hom(Z,X)$ by the equivalence relation which is generated by the simplicial homotopy relation. The set $\pi(Z,X)$ is the set of *simplicial homotopy classes* of maps from Z to X.

There is a contravariant set-valued functor which takes an object $[p] : Z \to X$ of $Triv/X$ to the set $\pi(Z,Y)$ of simplicial homotopy classes of maps between Z and Y. There is a function

$$\phi_h : \varinjlim_{[p]:Z \to X} \pi(Z,Y) \to [X,Y]$$

which is defined by sending the diagram of homotopy classes

$$X \xleftarrow{[p]} Z \xrightarrow{[f]} Y$$

to the morphism $f \cdot p^{-1}$ in the homotopy category. Observe that the simplicial homotopy relation preserves weak equivalences.

The colimit

$$\varinjlim_{[p]:Z\to X} \pi(Z,Y)$$

is the set of path components of a category $H_h(X,Y)$ whose objects are the pictures of simplicial homotopy classes

$$X \xleftarrow{[p]} Z \xrightarrow{[f]} Y,$$

such that $p : Z \to X$ is a hypercover, and whose morphisms are the commutative diagrams

$$\begin{tikzcd} & Z \arrow[dl, "{[p]}"'] \arrow[dr, "{[f]}"] \arrow[dd, "{[\theta]}"] & \\ X & & Y \\ & Z' \arrow[ul, "{[p']}"] \arrow[ur, "{[f']}"'] & \end{tikzcd} \tag{5.8}$$

in homotopy classes of maps. The map ϕ_h therefore has the form

$$\phi_h : \pi_0 H_h(X,Y) \to [X,Y]$$

The following result is the *Verdier hypercovering theorem*:

Theorem 5.41. *The function*

$$\phi_h : \pi_0 H_h(X,Y) \to [X,Y]$$

is a bijection if the simplicial presheaf Y is locally fibrant.

Remark 5.42. Theorem 5.41 is a generalization of the Verdier hypercovering theorem of [9, p.425] and [31], in which X is required to be locally fibrant. The statement of Theorem 5.41 first appeared, without proof, in [55].

There are multiple variants of the category $H_h(X,Y)$:

1) Write $H_h'(X,Y)$ for the category whose objects are pictures

$$X \xleftarrow{p} Z \xrightarrow{[f]} Y$$

where p is a hypercover and $[f]$ is a homotopy class of maps. The morphisms of $H_h'(X,Y)$ are diagrams

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "{[f]}"] \arrow[dd, "{[\theta]}"] & \\ X & & Y \\ & Z' \arrow[ul, "p'"] \arrow[ur, "{[f']}"'] & \end{tikzcd} \tag{5.9}$$

such that $[\theta]$ is a fibrewise homotopy class of maps over x, and $[f'][\theta] = [f]$ as simplicial homotopy classes. There is a functor

$$\omega : H'_h(X,Y) \to H_h(X,Y)$$

which is defined by the assignment $(p,[f]) \mapsto ([p],[f])$, and which sends the morphism (5.9) to the morphism (5.8).

2) Write $H''_h(X,Y)$ for the category whose objects are the pictures

$$X \xleftarrow{p} Z \xrightarrow{[f]} Y$$

where p is a hypercover and $[f]$ is a simplicial homotopy class of maps. The morphisms of $H''_h(X,Z)$ are commutative diagrams

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dd, "\theta"] \\ X & \\ & Z' \arrow[ul, "p'"] \end{tikzcd}$$

such that $[f' \cdot \theta] = [f]$. There is a canonical functor

$$H''_h(X,Y) \xrightarrow{\omega'} H'_h(X,Y)$$

which is the identity on objects, and takes morphisms θ to their associated fibrewise homotopy classes.

3) Let $h_{hyp}(X,Y)$ be the full subcategory of $h(X,Y)$ whose objects are the cocycles

$$X \xleftarrow{p} Z \xrightarrow{f} Y$$

with p a hypercover. There is a functor

$$\omega'' : h_{hyp}(X,Y) \to H''_h(X,Y)$$

which takes a cocycle (p,f) to the object $(p,[f])$.

Lemma 5.43. *Suppose that Y is locally fibrant. Then the inclusion functor*

$$i : h_{hyp}(X,Y) \subset h(X,Y)$$

is a homotopy equivalence.

Proof. Objects of the cocycle category $h(X,Y)$ can be identified with maps $(g,f) : Z \to X \times Y$ such that the morphism g is a weak equivalence, and morphisms of $h(X,Y)$ are commutative triangles in the obvious way. Maps of the form (g,f) have functorial factorizations

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "(g,f)"'] & V \arrow[d, "(p{,}g')"] \\ & X \times Y \end{tikzcd} \tag{5.10}$$

such that j is a sectionwise trivial cofibration and (p,g') is a sectionwise Kan fibration. It follows that (p,g') is a local fibration and the map p, or rather the composite

$$Z \xrightarrow{(p,g')} X \times Y \xrightarrow{pr} X,$$

is a local weak equivalence. The projection map pr is a local fibration since Y is locally fibrant, so the map p is also a local fibration, and hence a hypercover.

It follows that the assignment $(u,g) \mapsto (p,g')$ defines a functor

$$\psi' : h(X,Y) \to h_{hyp}(X,Y).$$

The weak equivalences j of (5.10) define homotopies $\psi' \cdot i \simeq 1$ and $i \cdot \psi' \simeq 1$.

Proof (Proof of Theorem 5.41). Write ψ for the composite functor

$$h(X,Y) \xrightarrow{\psi'} h_{hyp}(X,Y) \xrightarrow{\omega''} H''_h(X,Y) \xrightarrow{\omega'} H'_h(X,Y) \xrightarrow{\omega} H_h(x,y).$$

The composite function

$$\begin{aligned} \pi_0 h(X,Y) \xrightarrow{\psi'_*} \pi_0 h_{hyp}(X,Y) \xrightarrow{\omega''_*} \pi_0 H''_h(X,Y) \xrightarrow{\omega'_*} \pi_0 H'_h(X,Y) \\ \xrightarrow{\omega_*} \pi_0 H_h(X,Y) \xrightarrow{\phi_h} [X,Y] \end{aligned} \tag{5.11}$$

is the bijection ϕ of Theorem 5.34. The function ψ'_* is a bijection by Lemma 5.43, and the functions ω''_*, ω'_* and ω_* are surjective, as is the function ϕ_h. It follows that all of the functions which make up the string (5.11) are bijections.

The following corollary of the proof of Theorem 5.41 deserves independent mention:

Corollary 5.44. *Suppose that the simplicial presheaf Y is locally fibrant. Then the induced functions*

$$\pi_0 h_{hyp}(X,Y) \xrightarrow{\omega''_*} \pi_0 H''_h(X,Y) \xrightarrow{\omega'_*} \pi_0 H'_h(X,Y) \xrightarrow{\omega_*} \pi_0 H_h(X,Y)$$

are bijections, and all of these sets are isomorphic to the set $[X,Y]$ of morphisms $X \to Y$ in the homotopy category $\mathrm{Ho}(s/\mathbf{Pre}(\mathscr{C}))$.

The bijections of the path component objects in the statement of Corollary 5.44 with the set $[X,Y]$ all represent specific variants of the Verdier hypercovering theorem.

Remark 5.45. There is a relative version of Theorem 5.41, which holds for the model structures on slice category $A/s\mathbf{Pre}(\mathscr{C})$ which is induced from the injective model structure. Recall that the objects of this category are the simplicial presheaf maps $x : A \to X$, and the morphisms $f : x \to y$ are the commutative diagrams

$$\begin{tikzcd} & A \arrow[dl, "x"'] \arrow[dr, "y"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

In the induced model structure, the morphism $f : x \to y$ is a weak equivalence (respectively cofibration, fibration) if and only if the underlying map $f : X \to Y$ is a local weak equivalence (respectively cofibration, injective fibration) of simplicial presheaves.

In general, if $\mathbf{M}$ is a closed model category and A is an object of $\mathbf{M}$, then the slice category $A/\mathbf{M}$ inherits a model structure from $\mathbf{M}$ with the same definitions of weak equivalence, fibration and cofibration as above, and it is an easy exercise to show that this model structure exists.

The slice category $A/s\mathbf{Pre}(\mathscr{C})$ has a theory of cocycles by Theorem 5.34, and then the argument for Theorem 5.41 goes through as displayed above in the case where the target Y of the object $y : A \to Y$ is locally fibrant.

These observations apply, in particular, to give a Verdier hypercovering theorem for pointed simplicial presheaves. More detail can be found in [45].

In some respects, Lemma 5.43 is the "real" Verdier hypercovering theorem, although the result is a little awkward again because the cocycle categories in the statement might not be small. This situation is easily remedied by introducing cardinality bounds.

Suppose that α is an infinite cardinal such that the simplicial presheaves X and Y are α-bounded, and write $h_{hyp}(X,Y)_\alpha$ for full subcategory of the cocycle category $h(X,Y)$ on the cocycles

$$X \xleftarrow{p} Z \xrightarrow{f} Y$$

for which p is a hypercover and Z is α-bounded. Then $h_{hyp}(X,Y)_\alpha$ is a full subcategory of the category $h_\alpha(X,Y)$ of Proposition 5.36, and we have the following:

Theorem 5.46. *Suppose that α is an infinite cardinal such that $\alpha > |\mathrm{Mor}(\mathscr{C})|$. Suppose that the simplicial presheaves X and Y on the site $\mathscr{C}$ are α-bounded and that Y is locally fibrant. Then the inclusion $h_{hyp}(X,Y)_\alpha \subset h(X,Y)_\alpha$ induces a weak equivalence*

$$Bh_{hyp}(X,Y)_\alpha \xrightarrow{\simeq} Bh(X,Y)_\alpha.$$

Proof. The inclusion $h_{hyp}(X,Y)_\alpha \subset h(X,Y)_\alpha$ is a homotopy equivalence of small categories, by the same argument as for Lemma 5.43. In particular, the construction of the homotopy inverse functor

$$h(X,Y)_\alpha \to h_{hyp}(X,Y)_\alpha$$

from that proof respects cardinality bounds, by the assumptions on the size of the cardinal α.

Theorem 5.46 also leads to "hammock localization" results for simplicial presheaves. Suppose that the conditions for Theorem 5.46 hold in the following.

As in the proof of Lemma 5.39, write we_α/X for the category whose objects are all weak equivalences $U \to X$ with U α-bounded. The morphisms of we_α/X are the commutative diagrams

$$\begin{tikzcd} U \arrow[d] \arrow[dr, "\simeq"] & \\ U' \arrow[r, "\simeq"'] & X \end{tikzcd}$$

Suppose that the simplicial presheaf Z is injective fibrant, and consider the functor

$$\mathbf{hom}(\ ,Z) : (we_\alpha/X)^{op} \to s\mathbf{Set}$$

which is defined by the assignment

$$U \xrightarrow{\simeq} X \mapsto \mathbf{hom}(U,Z).$$

There is a canonical map

$$\underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z) \to B(we_\alpha/X)^{op}.$$

Since X is α-bounded, the category we_α/X has a terminal object, namely 1_X, so $B(we_\alpha/X)^{op}$ is contractible, while the diagram $\mathbf{hom}(\ ,Z)$ is a diagram of weak equivalences since Z is injective fibrant. It follows [24, IV.5.7] that the canonical map

$$\mathbf{hom}(X,Z) \to \underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z)$$

is a weak equivalence. At the same time, the horizontal simplicial set

$$\underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z)_n$$

is the nerve of the cocycle category $h(X,Z^{\Delta^n})_\alpha$ and is therefore weakly equivalent to $Bh(X,Z)_\alpha$, for all n, by Corollary 5.40. This means that the canonical map

$$Bh(X,Z)_\alpha \to \underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z)$$

is a weak equivalence.

We have proved the following:

Theorem 5.47. *Suppose that Z is injective fibrant and that X is α-bounded, where α is an infinite cardinal such that $\alpha > |\mathrm{Mor}(\mathscr{C})|$. Then the canonical maps*

$$Bh(X,Z)_\alpha \to \underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z) \leftarrow \mathbf{hom}(X,Z)$$

are weak equivalences.

The assertion that $Y \to Y^{\Delta^n}$ is a local weak equivalence holds for any locally fibrant object Y, and so we have the following:

Corollary 5.48. *Suppose that Y is locally fibrant and that $j : Y \to Z$ is an injective fibrant model in simplicial presheaves. Suppose that X is α-bounded. Then the simplicial set maps*

$$\begin{tikzcd} Bh(X,Y)_\alpha \arrow[r] & \underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Y) \arrow[d] & \\ & \underrightarrow{\mathrm{holim}}_{U \xrightarrow{\simeq} X} \mathbf{hom}(U,Z) & \mathbf{hom}(X,Z) \arrow[l] \end{tikzcd}$$

are weak equivalences.

Proof. The maps $Y \to Y^{\Delta^n} \to Z^{\Delta^n}$ are weak equivalences since Y and Z are locally fibrant. All maps

$$BH(X,Y)_\alpha \to Bh(X,Y^{\Delta^n})_\alpha \to Bh(X,Z^{\Delta^n})_\alpha$$

are therefore weak equivalences by Corollary 5.40.

Chapter 6
Localization theories

6.1 General theory

Suppose that $\mathscr{C}$ is a small Grothendieck site. Suppose that F is a fixed set of cofibrations $A \to B$ in the category $s\mathbf{Pre}(\mathscr{C})$ of simplicial presheaves on the site $\mathscr{C}$.

Throughout this chapter, we will assume that I is a simplicial presheaf on $\mathscr{C}$ with two disjoint global sections $0,1 : * \to I$. The object I will be called an *interval*, whether it looks like one or not. The examples that we are most likely to care about include the following:

1) the simplicial set Δ^1 with the two vertices $0,1 : * \to \Delta^1$,
2) $B\pi(\Delta^1)$ with the two vertices $0,1 : * \to \pi(\Delta^1)$ in the fundamental groupoid $\pi(\Delta^1)$ of Δ^1,
3) the affine line $\mathbb{A}^1$ over a scheme S with the rational points $0,1 : S \to \mathbb{A}^1$.

The basic idea behind the flavour of localization theory which will be presented here, is that one wants to construct, in a minimal way, a homotopy theory on simplicial presheaves for which the cofibrations are the monomorphisms, all of the maps in the set S become weak equivalences, and the interval object I describes homotopies.

Write

$$\square^n = I^{\times n}.$$

There are face inclusions

$$d^{i,\varepsilon} : \square^{n-1} \to \square^n,\ 1 \leq i \leq n,\ \varepsilon = 0,1,$$

with

$$d^{i,\varepsilon}(x_1,\dots,x_{n-1}) = (x_1,\dots,x_{i-1},\varepsilon,x_i,\dots,x_{n-1}).$$

The *boundary* $\partial\square^n$ and (i,ε)-*horn* $\sqcap^n_{i,\varepsilon}$ of $\square^n$ are defined, respectively, by

$$\partial\square^n = \cup_{i,\varepsilon}\, d^{i,\varepsilon}(\square^{n-1}),$$

and

$$\sqcap^n_{i,\varepsilon} = \cup_{(j,\gamma)\neq(i,\varepsilon)} d^{j,\gamma}(\square^{n-1}).$$

The interval I is used to define homotopies. A *naive homotopy* between maps $f,g : X \to Y$ is a commutative diagram

$$\begin{tikzcd} X \arrow[d, "0"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, "h"] & Y \\ X \arrow[u, "1"] \arrow[ur, "g"'] & \end{tikzcd} \tag{6.1}$$

Naive homotopies generate an equivalence relation: write

$$\pi(X,Y) = \pi_I(X,Y)$$

for the set of naive homotopy classes of maps $X \to Y$.

The class of F-*anodyne extensions* is the saturation of the set of inclusions $\Lambda(F)$ which consists of the maps

$$(C \times \square^n) \cup (D \times \sqcap^n_{(i,\varepsilon)}) \subset D \times \square^n \tag{6.2}$$

where $C \to D$ is a member of the set of generating cofibrations for $s\mathbf{Pre}(\mathscr{C})$, together with the maps

$$(A \times \square^n) \cup (B \times \partial\square^n) \subset B \times \square^n \tag{6.3}$$

with $A \to B$ in the set F. All F-anodyne extensions are cofibrations.

An F-*injective morphism* is a simplicial presheaf map $p : X \to Y$ which has the right lifting property with respect to all F-anodyne extensions, and a simplicial presheaf X is F-injective if the map $X \to *$ is an F-injective morphism.

The class of F-anodyne extensions includes all maps

$$(E \times \square^n) \cup (F \times \sqcap^n_{(i,\varepsilon)}) \subset F \times \square^n$$

which are induced by arbitrary cofibrations $E \to F$ of simplicial presheaves. It follows by a standard argument that, if Z is an F-injective object and X is an arbitrary simplicial presheaf, then the maps $f,g : X \to Z$ represent the same naive homotopy class in $\pi(X,Z)$ if and only if there is a single naive homotopy

$$\begin{tikzcd} X \arrow[d, "0"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, "h"] & Z \\ X \arrow[u, "1"] \arrow[ur, "g"'] & \end{tikzcd}$$

An F-*weak equivalence* is a map $f : X \to Y$ which induces a bijection

$$\pi(Y,Z) \xrightarrow{\cong} \pi(X,Z)$$

in naive homotopy classes of maps for all F-injective objects Z.

Observe that a map $f : X \to Y$ between F-injective objects is an F-weak equivalence if and only if it is a naive homotopy equivalence. In effect, there is a map $g : Y \to X$ such that $f \cdot g$ is naively homotopic to the identity 1_Y, and then $f \cdot g \cdot f$ and f are naively homotopic, so that $g \cdot f$ is naively homotopic to the identity 1_X.

A *cofibration* is again a monomorphism of simplicial presheaves, as for the injective model structure, and an F-*fibration* is a map which has the right lifting property with respect to all maps which are cofibrations and F-weak equivalences.

Lemma 6.1. *1) Suppose that $C \to D$ is an F-anodyne extension. Then the induced map*

$$(C \times \square^1) \cup (D \times \partial\square^1) \subset D \times \square^1 \tag{6.4}$$

is an F-anodyne extension.
2) All F-anodyne extensions are F-weak equivalences.

Proof. Show that that if $C \to D$ is in $\Lambda(F)$, then the induced map (6.4) is an F-anodyne extension. Statement 1) therefore holds for all generators $C \to D$ of the class of F-anodyne extensions, so it holds for all F-anodyne extensions, by a colimit argument.

Suppose that $i : C \to D$ is an F-anodyne extension and that Z is an injective object. Then the lifting exists in any diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "i"'] & Z \\ D \arrow[ur, dotted] & \end{tikzcd}$$

so that the map

$$i^* : \pi(D,Z) \to \pi(C,Z)$$

is surjective. If $f,g : D \to Z$ are morphisms such that there is a homotopy $h : C \times I \to Z$ between fi and gi, then the lifting exists in the diagram

$$\begin{tikzcd} (C \times \square^1) \cup (D \times \partial\square^1) \arrow[r, "{(h,(f,g))}"] \arrow[d] & Z \\ D \times \square^1 \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

by part 1), and the map H is a homotopy between f and g. It follows that the function

$$i^* : \pi(D,Z) \to \pi(C,Z)$$

is injective as well as surjective.

We shall prove the following:

Theorem 6.2. *Suppose that $\mathscr{C}$ is a small Grothendieck site, F is a set of cofibrations, and that I is some choice of interval. Then the simplicial presheaf category $s\mathbf{Pre}(\mathscr{C})$, with the cofibrations, F-weak equivalences and F-fibrations, has the structure of a cubical model category.*

The model structure of Theorem 6.2 is the *F-local model structure* for the category $s\mathbf{Pre}(\mathscr{C})$ of simplicial presheaves on the site $\mathscr{C}$.

The *cubical function complex* $\mathbf{hom}(X,Y)$ has n-cells defined to be the maps

$$X \times \square^n \to Y.$$

The assertion that the F-local model structure is cubical means that a cubical version of Quillen's simplicial model axiom **SM7** holds. In the present case, this means that

1) a map $A \to B$ which a cofibration and an F-weak equivalence induces an F-weak equivalence
$$(B \times \partial\square^n) \cup (A \times \square^n) \subset B \times \square^n$$
and
2) all cofibrations $C \to D$ induce F-weak equivalences
$$(D \times \sqcap^n_{(i,\varepsilon)}) \cup (C \times \square^n) \subset D \times \square^n.$$

Both statements are consequences of Lemma 6.1.

The model structure of Theorem 6.2 is automatically left proper, since all objects are cofibrant. There is a right properness assertion as well, which requires some extra conditions. This is proved in Theorem 6.19 below.

Theorems 6.2 and 6.19 are special cases of more general results, which can be found in [41]. Theorem 6.2 was originally proved by Cisinski [11], although he did not express it in the way which is displayed here. There is an older, more traditional approach to localization theory for simplicial presheaves which can be found in [23], but see also [27]. The big advantage of Theorem 6.2 and its proof is that an underlying model structure is not required: the set F can be any set of cofibrations, and in particular can be empty.

Note that the two maps $0,1 : * \to I$ are F-anodyne extensions, and therefore become weak equivalences in the F-local model structure, by Lemma 6.1.

The proof of Theorem 6.2 occupies much of this section. We begin by establishing a standard list of cardinality tricks.

Generally, suppose that G is some set of cofibrations of simplicial presheaves, and choose an infinite cardinal α such that $\alpha > |\mathrm{Mor}(\mathscr{C})|$, $\alpha > |G|$ and that $\alpha > |D|$ for all $C \to D$ in G.

Suppose that $\lambda > 2^\alpha$

Every $f : X \to Y$ has a functorial system of factorizations

$$\begin{tikzcd} X \arrow[r, "i_s"] \arrow[dr, "f"'] & E_s(f) \arrow[d, "f_s"] \\ & Y \end{tikzcd}$$

for $s < \lambda$ defined by the lifting property for maps in G, and which form the stages of a transfinite small object argument.

Specifically, given the factorization $f = f_s i_s$ form the pushout diagram

$$\begin{tikzcd} \bigsqcup_D C \arrow[r] \arrow[d] & E_s(f) \arrow[d] \\ \bigsqcup_D D \arrow[r] & E_{s+1}(f) \end{tikzcd}$$

where D is the set of all diagrams

$$\begin{tikzcd} C \arrow[r] \arrow[d, "i"'] & E_s(f) \arrow[d, "f_s"] \\ D \arrow[r] & Y \end{tikzcd}$$

with i in G. Then $f_{s+1} : E_{s+1}(f) \to Y$ is the obvious induced map. Set

$$E_t(f) = \varinjlim_{s<t} E_s(f)$$

at limit ordinals $t \leq \lambda$.

Then there is a functorial factorization

$$\begin{tikzcd} X \arrow[r, "i_\lambda"] \arrow[dr, "f"'] & E_\lambda(f) \arrow[d, "f_\lambda"] \\ & Y \end{tikzcd} \tag{6.5}$$

Also, f_λ has the right lifting property with respect to all $C \to D$ in G, and i_λ is in the saturation of G.

Write $\mathscr{L}(X) = E_\lambda(X \to *)$.

Lemma 6.3. *1) Suppose that $t \mapsto X_t$ is a diagram of simplicial presheaves, indexed by $\omega > 2^\alpha$. Then the map*

$$\varinjlim_{t<\omega} \mathscr{L}(X_t) \to \mathscr{L}(\varinjlim_{t<\omega} X_t)$$

is an isomorphism.

2) The functor $X \mapsto \mathscr{L}(X)$ preserves cofibrations.

3) Suppose that γ is a cardinal with $\gamma > \alpha$, and let $S_\gamma(X) =$ the subobjects of X having cardinality less than γ. Then the map

$$\varinjlim_{Y \in S_\gamma(X)} \mathscr{L}(Y) \to \mathscr{L}(X)$$

is an isomorphism.

4) If $|X| < 2^\mu$ where $\mu \geq \lambda$ then $|\mathscr{L}(X)| < 2^\mu$.

5) Suppose that U, V are subobjects of X. Then the natural map

$$\mathscr{L}(U \cap V) \to \mathscr{L}(U) \cap \mathscr{L}(V)$$

is an isomorphism.

Proof. It suffices to prove all statements with $\mathscr{L}(X)$ replaced by $E_1(X)$. There is a pushout diagram

$$\begin{tikzcd} \bigsqcup_G (C \times \hom(C,X)) \arrow[r] \arrow[d] & X \arrow[d] \\ \bigsqcup_G (D \times \hom(C,X)) \arrow[r] & E_1 X \end{tikzcd}$$

Then, in sections,

$$E_1 X = \bigsqcup_G ((D(a) - C(a)) \times \hom(C,X)) \sqcup X(a).$$

so 5) follows. The remaining statements are exercises.

We now return to the case of interest.

Lemma 6.4. *Every simplicial presheaf map $f : X \to Y$ has a functorial factorization*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where j is an F-anodyne extension and the map p is F-injective.

Proof. This is the case $G = \Lambda(F)$ of the factorization (6.5), with $Z = E_\lambda(f)$.

Suppose that α is an infinite cardinal such that $\alpha > |\Lambda(F)|$ and that $\alpha > |D|$ for all $C \to D$ in $\Lambda(F)$. Suppose that $\lambda > 2^\alpha$.

The following Lemma says that cofibrations which are F-weak equivalences satisfy a bounded cofibration condition:

Lemma 6.5. *Suppose given a diagram*

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r] & Y \end{tikzcd}$$

of cofibrations such that i is an F-weak equivalence and $|A| < 2^\lambda$. Then there is a subobject $B \subset Y$ with $A \subset B$ such that $|B| < 2^\lambda$ and $B \cap X \to B$ is an F-weak equivalence.

Proof. The proof is due to Cisinski. It is innovative in the sense that it uses nothing but naive homotopy (6.1).

The map $i_* : \mathscr{L}X \to \mathscr{L}Y$ is a cofibration (by Lemma 6.3), and is a naive homotopy equivalence of F-injective objects. There is a map $\sigma : \mathscr{L}Y \to \mathscr{L}Y$ such that $\sigma \cdot i_* \simeq 1$ via a naive homotopy $h : \mathscr{L}X \times I \to \mathscr{L}X$. Form the diagram

$$\begin{tikzcd} (\mathscr{L}Y \times \{0\}) \cup (\mathscr{L}X \times I) \arrow[r, "(\sigma,h)"] \arrow[d] & \mathscr{L}X \\ \mathscr{L}Y \times I \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

The other end $\mathscr{L}Y \times \{1\}$ of the homotopy H defines a map σ' such that $\sigma' \cdot i_* = 1$, and $i_*\sigma' \simeq i_*\sigma \simeq 1$. We can therefore assume that $\sigma \cdot i_* = 1$.

Suppose that $A_s \subset Y$ and $|A_s| < 2^\lambda$. Then $|\mathscr{L}A_s \times I| < 2^\lambda$ by Lemma 6.3. Also, there is a 2^λ-bounded subobject A_{s+1} such that $A_s \subset A_{s+1}$ and there is a diagram

$$\begin{tikzcd} \mathscr{L}A_s \times I \arrow[r] \arrow[d] & \mathscr{L}A_{s+1} \arrow[d] \\ \mathscr{L}Y \times I \arrow[r, "K"'] & \mathscr{L}Y \end{tikzcd}$$

where $K : i_*\sigma \simeq 1$.

This is the successor ordinal step in the construction of a system $s \mapsto A_s$ with $s < \lambda$ (recall that $\lambda > 2^\alpha$) and $A = A_0$. Let $B = \varinjlim_s A_s$. Then, by construction, B is 2^λ-bounded and the restriction of the homotopy K to $\mathscr{L}B \times I$ factors through the inclusion $j_* : \mathscr{L}B \to \mathscr{L}Y$.

There is a pullback

$$\begin{tikzcd} \mathscr{L}(B \cap X) \arrow[r, "j'"] \arrow[d, "i'"'] & \mathscr{L}X \arrow[d, "i_*"] \\ \mathscr{L}B \arrow[r, "j_*"'] & \mathscr{L}Y \end{tikzcd}$$

and $i_*\sigma(\mathscr{L}B) \subset \mathscr{L}B$. It follows that there is a map $\sigma' : \mathscr{L}B \to \mathscr{L}(B \cap X)$ such that $\sigma' \cdot i' = 1$. The homotopy K restricts to a homotopy $\mathscr{L}B \times I \to \mathscr{L}B$ (by construction), and this is a homotopy $i'\sigma' \simeq 1$.

A *F-trivial cofibration* is a map which is a cofibration and an F-weak equivalence. The next step in the proof of Theorem 6.2 is to show that the class of F-trivial cofibrations is closed under pushout.

Lemma 6.6. *Suppose given a diagram*

$$\begin{tikzcd} C \arrow[r, "f{,}g"] \arrow[d, "i"'] & E \\ D & \end{tikzcd}$$

where i is a cofibration, and suppose that there is a naive homotopy $h : C \times I \to E$ from f to g. Then $g_* : D \to D \cup_g E$ is an F-weak equivalence if and only if $f_* : D \to D \cup_f E$ is an F-weak equivalence.

Proof. There are pushout diagrams

$$\begin{tikzcd} C \arrow[r, "d_0"] \arrow[d, "i"'] & C \times I \arrow[r, "h"] \arrow[d, "i_*"] & E \arrow[d, "i_*"] \\ D \arrow[r, "d_{0*}"'] & D \cup_C (C \times I) \arrow[r, "h'"'] \arrow[d, "j"'] & D \cup_f E \arrow[d, "j_*"] \\ & D \times I \arrow[r, "h_*"'] & (D \times I) \cup_h E \end{tikzcd}$$

where the top composite is f. The maps d_{0*}, j and j_* are F-anodyne extensions. Thus $f_* = h' \cdot d_{0*}$ is an F-weak equivalence if and only if h' is an F-weak equivalence, and h' is an F-weak equivalence if and only if h_* is an F-weak equivalence. Thus, f_* is an F-weak equivalence if and only if h_* is an F-weak equivalence. Similarly, g_* is an F-weak equivalence if and only if h_* is an F-weak equivalence.

Lemma 6.7. *Suppose that the map $i : C \to D$ is an F-trivial cofibration. Then the cofibration*

$$(C \times I) \cup (D \times \{0,1\}) \to D \times I$$

is an F-weak equivalence.

Proof. The diagram

$$\begin{tikzcd} C \times \{0,1\} \arrow[r] \arrow[d] & D \times \{0,1\} \arrow[r] \arrow[d] & \mathscr{L}D \times \{0,1\} \arrow[d] \\ C \times I \arrow[r] & D \times I \arrow[r] & \mathscr{L}D \times I \end{tikzcd}$$

induces a diagram

$$\begin{tikzcd} (C \times I) \cup (D \times \{0,1\}) \arrow[r] \arrow[d] & (C \times I) \cup (\mathscr{L}D \times \{0,1\}) \arrow[d] \\ D \times I \arrow[r] & \mathscr{L}D \times I \end{tikzcd}$$

in which the horizontal maps are F-anodyne extensions, and hence F-weak equivalences.

There is a factorization

$$\begin{tikzcd} C \arrow[r, "i'"] \arrow[dr, "i"'] & D' \arrow[d, "p"] \\ & D \end{tikzcd}$$

where i' is an F-anodyne extension and the map p is both F-injective and an F-weak equivalence. In the induced diagram

$$\begin{tikzcd} (C \times I) \cup (\mathscr{L}D' \times \{0,1\}) \arrow[r] \arrow[d] & (C \times I) \cup (\mathscr{L}D \times \{0,1\}) \arrow[d] \\ \mathscr{L}D' \times I \arrow[r] & \mathscr{L}D \times I \end{tikzcd}$$

the top horizontal map is induced by the homotopy equivalence

$$\mathscr{L}D' \times \{0,1\} \to \mathscr{L}D \times \{0,1\},$$

and is therefore an F-equivalence by Lemma 6.6. The bottom horizontal map is also a homotopy equivalence. The left hand vertical map is an F-equivalence by comparison with the map

$$(C \times I) \cup (D' \times \{0,1\}) \to D' \times I$$

which is an F-anodyne extension.

Lemma 6.8. *Suppose that $j : C \to D$ is an F-trivial cofibration. Then every map $\alpha : C \to Z$ with Z F-injective extends to a map $D \to Z$.*

Proof. There is a homotopy $h : C \times I \to Z$ from α to a map $\beta \cdot j$ for some map $\beta : D \to Z$, and then the homotopy extends, via the diagram

$$\begin{tikzcd} (C \times I) \cup (D \times \{1\}) \arrow[r, "{(h,\beta)}"] \arrow[d] & Z \\ D \times I \arrow[ur, dotted, "H"'] \end{tikzcd}$$

since the vertical map is an F-anodyne extension. The desired extension $D \to Z$ is the restriction of H to $D \times \{0\}$.

Lemma 6.9. *The class of F-trivial cofibrations is closed under pushout.*

Proof. Suppose given a pushout diagram

$$\begin{array}{ccc} C & \longrightarrow & C' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ D & \longrightarrow & D' \end{array}$$

where the map j is an F-trivial cofibration.

Then the diagram

$$\begin{array}{ccc} (C\times I)\cup(D\times\{0,1\}) & \longrightarrow & (C'\times I)\cup(D'\times\{0,1\}) \\ \downarrow & & \downarrow \\ D\times I & \longrightarrow & D'\times I \end{array}$$

is a pushout. The left vertical map is an F-trivial cofibration by Lemma 6.7, and therefore has the left lifting property with respect to the map $Z \to *$ by Lemma 6.8. Thus, if two maps $f,g : D' \to Z$ restrict to homotopic maps on C', then $f \simeq g$.

Every F-fibration is an F-injective map, since every F-anodyne extension is an F-trivial cofibration by Lemma 6.1. This observation has the following partial converse:

Lemma 6.10. *Suppose that the map $p : X \to Y$ is F-injective and that Y is an F-injective object. Then the map p is an F-fibration.*

Proof. Suppose given a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\beta]{} & Y \end{array} \tag{6.6}$$

where i is an F-trivial cofibration. Then there is a map $\theta : B \to X$ such that $\theta\cdot i = \alpha$ since X is F-injective.

The constant homotopy $A\times I \xrightarrow{pr} A \xrightarrow{\alpha} X$ extends to a homotopy $h : B\times I \to Y$ as in the diagram

$$\begin{tikzcd} (A\times I)\cup(B\times\{0,1\}) \arrow[r, "{(p\alpha pr_A,(\beta,p\theta))}"] \arrow[d] & Y \\ B\times I \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

since the vertical map is an F-trivial cofibration (Lemma 6.7) and Y is F-injective. It follows that there is a homotopy

$$\begin{tikzcd} A \times I \arrow[r, "\alpha pr_A"] \arrow[d, "i \times i"'] & X \arrow[d, "p"] \\ B \times I \arrow[r, "h"'] & Y \end{tikzcd}$$

from the original diagram to a diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "p\theta"'] \arrow[ur, dotted, "\theta"] & Y \end{tikzcd}$$

The lifting in the diagram

$$\begin{tikzcd} (A \times I) \cup B \arrow[r, "(\alpha pr_A{,}\theta)"] \arrow[d] & X \arrow[d, "p"] \\ B \times I \arrow[r, "h"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

restricts to the required lifting for the original diagram (6.6).

Corollary 6.11. *Every F-injective object is F-fibrant.*

Lemma 6.12 (CM4). *Suppose that $p : X \to Y$ is an F-fibration and an F-weak equivalence. Then p has the right lifting property with respect to all cofibrations.*

Proof. Suppose first that Y is F-injective. Then p is a naive homotopy equivalence, and has a section $\sigma : Y \to X$.

The map σ is an F-trivial cofibration so the lift exists in the diagram

$$\begin{tikzcd} (Y \times I) \cup (X \times \{0,1\}) \arrow[rr, "(\sigma \cdot pr{,}(1_X{,}\sigma \cdot p))"] \arrow[d] & & X \arrow[d, "p"] \\ X \times I \arrow[r, "p \times 1"'] \arrow[urr, dotted, "H"] & Y \times I \arrow[r, "pr"'] & Y \end{tikzcd}$$

since the left vertical map is an F-weak equivalence by Lemma 6.7. It follows that the identity diagram on $p : X \to Y$ is naively homotopic to the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma \cdot p"] \arrow[d, "p"'] & X \arrow[d, "p"] \\ Y \arrow[r, "1"'] \arrow[ur, dotted, "\sigma"] & Y \end{tikzcd}$$

Thus, any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

is naively homotopic to a diagram which admits a lifting. It follows that the map p has the right lifting property with respect to all cofibrations.

If Y is not F-injective, form the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & Z \arrow[d, "q"] \\ Y \arrow[r, "j_Y"'] & \mathscr{L}(Y) \end{tikzcd}$$

where j is an F-anodyne extension and q is F-injective. Then q is an F-fibration by Lemma 6.9 and is an F-weak equivalence by Lemma 6.1, so that q has the right lifting property with respect to all cofibrations by the previous paragraph.

Factorize the map $X \to Y \times_{\mathscr{L}(Y)} Z$ as

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & W \arrow[d, "\pi"] \\ & Y \times_{\mathscr{L}(Y)} Z \end{tikzcd}$$

where π has the right lifting property with respect to all cofibrations and i is a cofibration. Write q_* for the induced map $Y \times_{\mathscr{L}(Y)} Z \to Y$. Then the composite $q_*\pi$ has the right lifting property with respect to all cofibrations and is therefore a naive homotopy equivalence. The cofibration i is therefore an F-weak equivalence, and it follows that the lifting exists in the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Z \arrow[r, "q_*\pi"'] \arrow[ur] & Y \end{tikzcd}$$

so that p is a retract of a map which has the right lifting property with respect to all cofibrations.

Corollary 6.13. *A map $p : X \to Y$ is an F-fibration and an F-weak equivalence if and only if it has the right lifting property with respect to all cofibrations.*

We are now ready to finish the proof of Theorem 6.2.

Proof (Proof of Theorem 6.2). The cofibration/trivial fibration factorization statement of **CM5** is a consequence of Corollary 6.13, along with the corresponding factorization for the injective model structure.

A map $p : X \to Y$ is an F-fibration if and only if it has the right lifting property with respect to all α-bounded F-trivial cofibrations for a suitable choice of infinite cardinal α — this is a consquence of Lemma 6.5. The trivial cofibration/fibration factorization statement follows by a standard small object argument. We need to know that trivial cofibrations are closed under pushout, but this is Lemma 6.9.

All simplicial presheaves are cofibrant for the present model structure. Left properness therefore follows from general nonsense about categories of cofibrant objects — see [24, II.8.5].

Example 6.14 (Homotopy theory of simplicial presheaves). Suppose that F is a generating set of trivial cofibrations $A \to B$ for the inductive model structure on $s\mathbf{Pre}(\mathscr{C})$, and that $I = \Delta^1$ is the standard interval.

An injective model $j : X \to \mathscr{L}(X)$ is an injective fibrant model since all F-anodyne extensions are trivial cofibrations for the injective structure and all F-injective objects are injective fibrant. Thus, every F-weak equivalence is a local weak equivalence. If $f : X \to Y$ is a local weak equivalence, then $\mathscr{L}(X) \to \mathscr{L}(Y)$ is a local weak equivalence between injective fibrant models, and is therefore a standard homotopy equivalence; it follows that f is an F-weak equivalence.

Example 6.15 (The f-local theory, for a cofibration $f : A \to B$). Suppose that F consists of a generating set of trivial cofibrations $C \to D$ *as well as* all cofibrations

$$(B \times C) \cup (A \times D) \xrightarrow{(f,j)} B \times D$$

which are induced by a set of generators $j : C \to D$ of the class of cofibrations of $s\mathbf{Pre}(\mathscr{C})$. Let $I = \Delta^1$ be the standard interval. The corresponding F-local model structure is the *f-local model structure* for simplicial presheaves [23], [38], [41]. One usually says that an F-weak equivalence is an f-weak equivalence and that F-fibrations are f-fibrations in this case.

Every local weak equivalence is an f-weak equivalence. It follows by an induction on skeleta that the functor $X \mapsto X \times K$ preserves f-weak equivalences for all simplicial sets K, and that a map

$$(B \times K) \cup (A \times L) \subset B \times L$$

which is induced by a cofibration $i : A \to B$ and an inclusion $i : K \to L$ of simplicial sets is an f-equivalence if either i is an f-equivalence or j is a trivial cofibration. It follows in particular that the f-local structure has a closed simplicial model structure, with respect to the standard function complex $\mathbf{hom}(X,Y)$. It also follows that a map $g : X \to Y$ is an f-weak equivalence if and only if it induces a weak equivalence

$$g^* : \mathbf{hom}(Y,Z) \to \mathbf{hom}(X,Z)$$

for all f-fibrant simplicail presheaves Z.

Example 6.16 (Motivic homotopy theory). Suppose that S is a scheme of finite dimension (typically a field), and let $(Sm|_S)_{Nis}$ be the category of smooth schemes of finite type over S, equipped with the Nisnevich topology. The motivic model structure on $s\mathbf{Pre}(Sm|_S)_{Nis}$ can be constructed in two ways:

a) Let F consist of the generating set of the trivial cofibrations for the injective model structure on $s\mathbf{Pre}(Sm|_S)_{Nis}$, together with all maps

$$(C \times \mathbb{A}^1) \cup D \subset D \times \mathbb{A}^1$$

which are induced by the 0-section $f : * \to \mathbb{A}^1$ and generators $C \to D$ of the class of cofibrations, and let $I = \Delta^1$. This is the f-local theory associated to the 0-section $f : * \to \mathbb{A}^1$.

b) Let F be the generating set of trivial cofibrations for the injective model structure on $s\mathbf{Pre}(Sm|_S)_{Nis}$ and let $I = \mathbb{A}^1$ with the global sections $0, 1 : * \to \mathbb{A}^1$.

It is an exercise to show that the two model structures coincide: every F-anodyne extension of one structure is a trivial cofibration of the other, and so the two structures have same injective objects. It follows that the two classes of weak equivalences coincide.

The motivic model structure is called the $\mathbb{A}^1$-model structure in [55]. Strictly speaking, the Morel-Voevodsky model structure is on the category of simplicial sheaves on the smooth Nisnevich site, but the model structures for simplicial sheaves and simplicial presheaves are Quillen equivalent by the usual argument [38]. There are many other models for motivic homotopy theory, including model structures on presheaves and sheaves (not simplicial!) on the smooth Nisnevich site [38], and all the models arising from test categories [41].

Example 6.17 (Quasi-categories). The quasi-category model structure on the category $s\mathbf{Set}$ of simplicial sets is the model structure given by the theorem for the set F of *inner anodyne extensions*

$$\Lambda^n_k \subset \Delta^n,\ 0 < k < n,$$

and the interval $I = B\pi(\Delta^1)$. Here, $\pi(\Delta^1)$ is the fundamental groupoid on the simplex Δ^1; it is the trivial groupoid with objects consisting of the set $\{0, 1\}$ of vertices. A *quasi-category* is a simplicial set X such that the map $X \to *$ with respect to all inner anodyne extensions.

The following is an analog (and a partial consequence) of Lemma 5.17, for the f-local theory of Example 6.15:

Lemma 6.18. *1) Suppose that $g : X \to Y$ is an f-weak equivalence and that E is a simplicial presheaf. Then the map*

$$g \times 1 : X \times E \to Y \times E$$

is an f-weak equivalence.

2) Suppose that $i : A \to B$ and $j : C \to D$ are cofibrations of $s\mathbf{Pre}(\mathscr{C})$. Then the induced cofibration

$$(B \times C) \cup (A \times D) \to B \times D$$

is an f-weak equivalence if either i or j is an f-weak equivalence.

Proof. All maps

$$(B \times C) \cup (A \times D) \xrightarrow{(f,j)} B \times D$$

which are induced by cofibrations $j : C \to D$ are f-weak equivalences. Thus, the cofibration $j \times 1 : C \times E \to D \times E$ induces an f-equivalence

$$((B \times C) \cup (A \times D)) \times E \xrightarrow{(f,j)\times 1} (B \times D) \times E.$$

It follows that if a cofibration $j : C' \to D'$ is in the set F, then the induced map

$$j \times 1 : C' \times E \to D' \times E$$

is an f-equivalence.

All cofibrations (6.2) are local weak equivalences since $I = \Delta^1$, and so all induced cofibrations

$$((C \times \square^n) \cup (D \times \sqcap^n_{(i,\varepsilon)})) \times E \subset (D \times \square^n) \times E$$

are local weak equivalences by Lemma 5.17. All cofibrations (6.3) induce f-equivalences

$$((A \times \square^n) \cup (B \times \partial\Delta^n)) \times E \subset (B \times \square^n) \times E$$

since all morphisms $A \to B$ in the set F induce f-equivalences $A \times E \to B \times E$ by the argument in the first paragraph.

It follows that the functor $X \mapsto X \times E$ takes F-anodyne extensions to f-equivalences.

If Z is f-fibrant, the lifting therefore exists in every diagram

$$\begin{tikzcd} C \times E \arrow[r] \arrow[d, "j\times 1"'] & Z \\ D \times E \arrow[ur, dotted] & \end{tikzcd}$$

for every F-anodyne extension $j : C \to D$, and it follows that the internal function complex $\mathbf{Hom}(E,Z)$ (see the discussion of this object in Section 5.2) is f-fibrant by Lemma 6.18.

A simplicial presheaf map $h : V \to W$ is an f-equivalence if and only if it induces a weak equivalence of simplicial sets

$$h^* : \mathbf{hom}(W,Z) \to \mathbf{hom}(V,Z)$$

for all f-fibrant objects Z. It follows that the f-equivalence g induces a weak equivalence

$$g^* : \mathbf{hom}(Y,\mathbf{Hom}(E,Z)) \to \mathbf{hom}(X,\mathbf{Hom}(E,Z)).$$

This simplicial set map is isomorphic to the map

$$(g \times 1)^* : \mathbf{hom}(Y \times E,Z) \to \mathbf{hom}(X \times E,Z).$$

Thus, $(g \times 1)^*$ is a weak equivalence of simplicial sets for all f-equivalences g and all f-fibrant objects Z, and so $g \times 1$ is an f-weak equivalence.

6.2 Properness

We already know that all of the F-local model structures of Theorem 6.2 are left proper, since all objects in these structures are cofibrant.

The following result gives a condition for an F-local structure to be right proper. It is a special case of Theorem 4.18 of [41], but the proof which is given here is new.

One of the outcomes, which appears in Corollary 6.21 below, is the Morel-Voevodsky result that the standard motivic model structures are proper.

Theorem 6.19. *In the F-local model structure of Theorem 6.2, suppose that all cofibrations in the set F pull back to F-weak equivalences along all F-fibrations $p : X \to Z$ with Z F-fibrant. Then the F-local model structure on $s\mathbf{Pre}(\mathscr{C})$ is proper.*

The condition in the statement of Theorem 6.19 means that, in every diagram

$$\begin{tikzcd}
A \times_Y X \arrow[r, "i_*"] \arrow[d] & B \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
A \arrow[r, "i"'] & B \arrow[r] & Y
\end{tikzcd}$$

with p an f-fibration such that Y is f-fibrant, if i is a member of f then the map i_* is an F-weak equivalence.

Proof (Proof of Theorem 6.19). Write $\mathscr{F}$ for the class of all F-fibrations which preserve F-weak equivalences under pullback. Suppose that we can show that all F-fibrations $Z \to W$ with F-fibrant base W are in $\mathscr{F}$.

Let $p : X \to Y$ be an F-fibration, and form the diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "p"'] & Z \arrow[d, "q"] \\
Y \arrow[r, "j"'] & \mathscr{L}(Y)
\end{tikzcd}$$

where j is an anodyne extension and $\mathscr{L}(Y)$ is fibrant, i is an F-trivial cofibration and q is an F-fibration. The induced map

$$\begin{tikzcd}
X \arrow[rr, "i_*"] \arrow[dr, "p"'] & & Y \times_{\mathscr{L}(Y)} Z \arrow[dl, "q_*"] \\
& Y &
\end{tikzcd}$$

is an F-trivial cofibration between F-fibrant objects of $s\mathbf{Pre}(\mathscr{C})/Y$, and so there is a map $r : q_* \to p$ such that $r \cdot i_* = 1$. It follows that p is a retract of a map q_*; the map q_* is in $\mathscr{F}$, so that p is a member of $\mathscr{F}$.

The projection map $pr : D \times \square^n \to D$ pulls back to an F-equivalence along all F-fibrations. Suppose that $p : X \to D \times \square^n$ is an F-fibration, and form the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "p"'] & Z \arrow[d, "q"] \\ D \times \square^n \arrow[r, "pr"'] & D \end{tikzcd}$$

where i is an F-trivial cofibration and q is an F-fibration. Then the induced map

$$\begin{tikzcd} X \arrow[rr, "i_*"] \arrow[dr, "p"'] & & Z \times \square^n \arrow[dl, "q\times 1"] \\ & D \times \square^n & \end{tikzcd}$$

is an F-trivial cofibration of F-fibrant objects in $s\mathbf{Pre}(\mathscr{C})/(D \times \square^n)$, so that p is a retract of $q \times 1$. It follows that $(1 \times j)_*$ in the pullback diagram

$$\begin{tikzcd} (D \times \sqcap^n_{(i,\varepsilon)}) \times_{(D\times\square^n)} X \arrow[r, "(1\times j)_*"] \arrow[d] & X \arrow[d, "p"] \\ D \times \sqcap^n_{(i,\varepsilon)} \arrow[r, "1\times j"'] & D \times \square^n \end{tikzcd}$$

is an F-weak equivalence. Here, $j : \sqcap^n_{(i,\varepsilon)} \to \square^n$ is the usual inclusion. Similarly, the map $D \to D \times \square^n$ which is defined by the inclusion $* \to \square^n$ of a vertex induces a weak equivalence $D \times_{D\times\square^n} X \to X$ after pullback along the map p.

It follows that all maps

$$1 \times j : D \times \sqcap^n_{(i,\varepsilon)} \to D \times \square^n$$

pull back to F-weak equivalences along all F-fibrations. It also follows that the cofibrations

$$(C \times \square^n) \cup (D \times \sqcap^n_{(i,\varepsilon)}) \to D \times \square^n$$

which are induced by cofibrations $C \to D$ pull back to F-weak equivalences along all F-fibrations.

Write $\mathscr{W}$ for the class of all maps $f : U \to V$ which pull back to F-weak equivalences along all F-fibrations $X \to Y$ with F-fibrant base Y. All cofibrations $A \to B$ of the set F are in $\mathscr{W}$, by assumption.

Then all maps

$$f \times 1 : A \times \square^n \to B \times \square^n$$

with $f \in F$ are in the class $\mathscr{W}$ since all maps $A \to A \times \square^n$ which are defined by inclusion of vertices of $\square^n$ are in $\mathscr{W}$, by the the first paragraphs of the proof.

It follows by induction on n, by using comparisons of pushout diagrams

$$\begin{tikzcd} C \times \partial\square^{n-1} \arrow[r] \arrow[d] & C \times \sqcap^n_{(i,\varepsilon)} \arrow[d] \\ C \times \square^{n-1} \arrow[r] & C \times \partial\square^n \end{tikzcd}$$

that all morphisms $f \times 1 : A \times \partial\square^n \to B \times \partial\square^n$ which are induced by morphisms $A \to B$ of F are in $\mathscr{W}$. It also follows that all maps

$$(A \times \square^n) \cup (B \cup \partial\square^n) \subset B \times \square^n$$

which are induced by cofibrations $A \to B$ of F are in the class $\mathscr{W}$.

The class $\mathscr{W}$ is closed under retractions and transfinite compositions as well as pushout, so that all F-anodyne extensions are in $\mathscr{W}$.

Suppose that $p : X \to Y$ is an F-fibration with F-fibrant base Y, and consider a diagram

$$\begin{tikzcd} & & X \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \arrow[r, "\beta"'] & Y \end{tikzcd}$$

where f is an F-weak equivalence. Then there is a diagram

$$\begin{tikzcd} & & X \arrow[d, "p"] \\ A \arrow[r, "f"] \arrow[d, "j_A"'] & B \arrow[r, "\beta"] \arrow[d, "j_B"'] & Y \\ \mathscr{L}(A) \arrow[r, "f_*"'] & \mathscr{L}(B) \arrow[ur] & \end{tikzcd}$$

where j_A and j_B are F-anodyne extensions taking values in F-fibrant objects, and f_* is an F-weak equivalence. It follows that f pulls back to an F-weak equivalence along p if and only if the map f_* does so.

The object $\mathscr{L}(B)$ is F-fibrant, so f_* has a factorization

$$\begin{tikzcd} \mathscr{L}(A) \arrow[r, "j"] \arrow[dr, "f_*"'] & W \arrow[d, "q"] \\ & \mathscr{L}(B) \end{tikzcd}$$

such that j is an F-anodyne extension and q is an F-trivial fibration, by Lemma 6.10. The F-trivial fibration q pulls back to an F-weak equivalence along p by formal nonsense, and the F-anodyne extension j pulls back to a F-weak equivalence along p by the last paragraph.

All F-fibrations with F-fibrant base therefore preserve F-weak equivalences under pullback, so that the same is therefore true for all F-fibrations.

Corollary 6.20. *Suppose that $f : * \to A$ is a global section of a simplicial presheaf A on a small site $\mathscr{C}$. Then the f-local model structure on $s\mathbf{Pre}(\mathscr{C})$ is proper.*

Proof. It suffices, by Theorem 6.19, to show that the map $f : * \to I$ pulls back to an f-weak equivalence along all f-fibrations.

The idea of proof already appears in the proof of Theorem 6.19. Suppose that $p : X \to I$ is an f-fibration, and form the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & Z \arrow[d] \\ I \arrow[r] & * \end{tikzcd}$$

where j is an f-anodyne extension and Z is f-fibrant. Then the map p is a retract of the projection $pr : I \times Z \to I$ and the map f pulls back to an f-weak equivalence (actually an f-anodyne extension) $f \times 1 : * \times Z \to I \times Z$ along the map pr. It follows that the induced map $p^{-1}(f) \to X$ is an f-weak equivalence.

Corollary 6.21. *The motivic model structure on the category $s\mathbf{Pre}(Sm|_S)_{Nis}$ of simplicial presheaves on the smooth Nisnevich site is proper.*

6.3 Intermediate model structures

Throughout this section, suppose that $\mathbf{M}$ is a model structure on the category $s\mathbf{Pre}(\mathscr{C})$ for which the cofibrations are the monomorphisms. Suppose that every local weak equivalence is a weak equivalence of $\mathbf{M}$. Every object is cofibrant in the model structure $\mathbf{M}$, so that $\mathbf{M}$ is a left proper model structure.

It also follows that the class of weak equivalences of $\mathbf{M}$ is closed under products with simplicial sets. In effect, if $f : X \to Y$ is a weak equivalence of $\mathbf{M}$ then each induced map

$$f \times 1 : X \times \Delta^n \to Y \times \Delta^n$$

is locally equivalent and hence weakly equivalent to f in $\mathbf{M}$, and is therefore a weak equivalence of $\mathbf{M}$. It follows by an induction of skeleta (which involves the left properness of $\mathbf{M}$) that the map

$$f \times 1 : X \times K \to Y \times K$$

is a weak equivalence of $\mathbf{M}$ for each simplicial set K.

The standard function complexes $\mathbf{hom}(X,Y)$ therefore gives $\mathbf{M}$ the structure of a simplicial model category.

Examples include all cases of Theorem 6.2 for which the set of cofibrations F includes a set of generators for the trivial cofibrations for the injective model structure, so that the includes all localizations of the injective model structure, for whatever choice of interval I. These include all f-localizations (Example 6.15), and hence the standard motivic model structures (Example 6.16).

Recall from Section 1.4 that there is a *projective* model structure on $s\mathbf{Pre}(\mathscr{C})$, for which the fibrations are sectionwise Kan fibrations and the weak equivalences are sectionwise weak equivalences.

The cofibrations for this theory are the projective cofibrations, and this class of maps has a generating set S_0 consisting of all maps $L_U(\partial\Delta^n) \to L_U(\Delta^n)$, $n \geq 0$, $U \in \mathscr{C}$.

Write $\mathbf{C}_P$ for the class of projective cofibrations, and write $\mathbf{C}$ for the full class of cofibrations (which are the simplicial presheaf monomorphisms). Every projective cofibration is a cofibration, so there is a relation $\mathbf{C}_P \subset \mathbf{C}$.

Let S be any *set* of cofibrations which contains S_0. Let $\mathbf{C}_S$ be the saturation of the set of all cofibrations of the form

$$(B \times \partial\Delta^n) \cup (A \times \Delta^n) \subset B \times \Delta^n$$

which are induced by members $A \to B$ of the set S. The *saturation* of a collection of cofibrations [24, II.6.3] is the smallest class of cofibrations containing the list above, which contains all isomorphisms, and is closed under pushout, retracts, disjoint union, composition and transfinite composition. Say that $\mathbf{C}_S$ is the class of *S-cofibrations*.

An *S-fibration* is a map $p : X \to Y$ of simplicial presheaves which has the right lifting property with respect to all S-cofibrations which are weak equivalences of $\mathbf{M}$. Observe that every fibration of $\mathbf{M}$ is an S-fibration.

Theorem 6.22. *Let* $\mathbf{M}$ *be a model structure on the category* $s\mathbf{Pre}(\mathscr{C})$ *of simplicial presheaves for which the cofibrations are the monomorphisms, and suppose that every local weak equivalence is a weak equivalence of* $\mathbf{M}$*. Then the category* $s\mathbf{Pre}(\mathscr{C})$*, together with the classes of S-cofibrations, weak equivalences of* $\mathbf{M}$*, and S-fibrations, satisfies the axioms for a left proper closed simplicial model category.*

Proof. The axioms **CM1**, **CM2** and **CM3** are trivial to verify.

Any $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where $j \in \mathbf{C}_S$ and p has the right lifting property with respect to all members of $\mathbf{C}_S$. Then p is an S-fibration and is a sectionwise weak equivalence. The map p is therefore a weak equivalence of $\mathbf{M}$.

The map f also has a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & W \arrow[d, "q"] \\ & Y \end{tikzcd}$$

for which q is a fibration of $\mathbf{M}$ and i is a trivial cofibration of $\mathbf{M}$. Then q is an S-fibration. Factorize the map i as $i = p \cdot j$ where $j \in \mathbf{C}_S$ and p is an S-fibration and a weak equivalence of $\mathbf{M}$ (as above). Then j is a weak equivalence of $\mathbf{M}$, so $f = (qp) \cdot j$ factorizes f as an S-fibration following a map which is an S-cofibration and a weak equivalence of $\mathbf{M}$.

We have therefore proved the factorization axiom **CM5**.

It is an exercise to prove **CM4**. One shows that if $p : X \to Y$ is an S-fibration and a weak equivalence of $\mathbf{M}$, then p is a retract of a map which has the right lifting property with respect to all S-cofibrations.

Suppose that $j : K \to L$ is a cofibration of simplicial sets. The collection of all cofibrations $i : C \to D$ of simplicial presheaves such that the induced map

$$(D \times K) \cup (C \times L) \to D \times L \tag{6.7}$$

is and S-cofibration is saturated, and contains all generators

$$(B \times \partial\Delta^n) \cup (A \times \Delta^n) \subset B \times \Delta^n$$

of the class $\mathbf{C}_S$. It follows that the map (6.7) is an S-cofibration if $i : C \to D$ is an S-cofibration. This map is a weak equivalence of $\mathbf{M}$ if either i is a weak equivalence of $\mathbf{M}$ or j is a weak equivalence of simplicial sets. The model structure of the statement of the Theorem is therefore a simplicial model structure.

The left properness of this model structure is a consequence of the left properness for the ambient model category $\mathbf{M}$.

Example 6.23. The case $S = S_0$ for Theorem 6.22, and where $\mathbf{M}$ is the injective model structure on $s\mathbf{Pre}(\mathscr{C})$, gives the local projective structure of Blander [5] for simplicial presheaves on $\mathscr{C}$. The local projective structure, for the site of smooth S-schemes with the Nisnevich topology, has seen extensive applications.

If $\mathbf{M}$ is still the injective model structure on $s\mathbf{Pre}(\mathscr{C})$, but the set of cofibrations S is allowed to vary, then Theorem 6.22 gives the *intermediate model structures* for simplicial presheaves of [43].

All intermediate model structures are right proper, because all S-fibrations are local fibrations and pullbacks of local weak equivalences along local fibrations are local weak equivalences by Lemma 4.30.

Example 6.24. Let $(Sm/T)_{Nis}$ be the Nisnevich site of smooth schemes over a scheme T, and let $\mathbf{M}$ be the motivic model structure on the category of simplicial presheaves on this site (Example 6.16). The case $S = S_0$ of Theorem 6.22 for the motivic model structure $\mathbf{M}$ gives the *projective motivic model structure* for $s\mathbf{Pre}((Sm|_T)_{Nis})$ — see also [65]. Theorem 6.22 also gives a large collection of other motivic model structures which are intermediate between the projective and standard motivic model structures.

The model structure of Theorem 6.22 is cofibrantly generated, under an extra assumption that is satisfied in the usual examples. This was proved for the original intermediate model structures of Example 6.23 by Beke [4], whose method was to verify a solution set condition. Beke's argument can be deconstructed as in [43] to give a basic and useful trick for verifying cofibrant generation in the presence of some kind of cardinality calculus. That trick is reprised here, in the proof of Lemma 6.25 below.

The proof of Lemma 6.25 requires the assumption that the model structure $\mathbf{M}$ satisfies a bounded cofibration condition. This means that there is an infinite cardinal α such that, given a diagram

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r, "j"'] & Y \end{tikzcd}$$

such that i and j are cofibrations, i is an $\mathbf{M}$-trivial cofibration and A is α-bounded, there is an α-bounded subobject B of Y which contains A such that the induced cofibration $B \cap X \to M$ is a weak equivalence of $\mathbf{M}$.

An $\mathbf{M}$-trivial cofibration is a cofibration which is a weak equivalence of $\mathbf{M}$.

All model structures given by Theorem 6.2 satisfy a bounded cofibration condition, by Lemma 6.5. A bounded cofibration condition for the injective model structure is proved independently in Lemma 5.2.

We shall also assume that $\alpha > |\mathrm{Mor}(\mathscr{C})|$, and that $|D| < \alpha$ for all members $C \to D$ of the set of cofibrations generating $\mathbf{C}_S$.

Every α-bounded $\mathbf{M}$-trivial cofibration $i : A \to B$ has a factorization

$$\begin{tikzcd} A \arrow[r, "j_i"] \arrow[dr, "i"'] & C \arrow[d, "p_i"] \\ & B \end{tikzcd}$$

such that j_i is an S-cofibration, p_i is an S-fibration and both maps are weak equivalences of $\mathbf{M}$. Write I for the set of all S-cofibrations j_i which are constructed in this way.

Lemma 6.25. *Suppose that, in addition to the assumption of Theorem 6.22, the model structure* $\mathbf{M}$ *on the category* $s\mathbf{Pre}(\mathscr{C})$ *satisfies a bounded cofibration condition. Then the members of the set I generate the class of trivial S-cofibrations, and the S-model structure is cofibrantly generated.*

Proof. Suppose given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

such that i is an α-bounded member of $\mathbf{C}_S$ and f is a weak equivalence of $\mathbf{M}$. Then, since B is α-bounded, this diagram has a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[r] \arrow[d, "j"] & X \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

where j is a member of the set of S-cofibrations I.

In effect, by factorizing $f = p \cdot u$ where u is an $\mathbf{M}$-trivial cofibration and q is an $\mathbf{M}$- trivial fibration, we can assume that f is an $\mathbf{M}$-trivial cofibration. The bounded cofibration property then implies that there is a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[r] \arrow[d, "v"] & X \arrow[d, "f"] \\ B \arrow[r] & F \arrow[r] & Y \end{tikzcd}$$

as above with v an α-bounded trivial cofibration. Factorize $v = p_v j_v$ as above. The object F is a suitable extension of the image of B in Y, which image is α-bounded. Then p_v is a trivial S-fibration and therefore has the right lifting property with respect to i, and j_v is the desired member of the set I.

Every trivial S-cofibration $j : A' \to B'$ has a factorization

$$\begin{tikzcd} A' \arrow[r, "\beta"] \arrow[dr, "j"'] & C' \arrow[d, "q"] \\ & B' \end{tikzcd}$$

such that β is an S-cofibration in the saturation of the set I and q has the right lifting property with respect to all members of I. Then q is also a weak equivalence of $\mathbf{M}$, and therefore has the right lifting property with respect to all members of the class $\mathbf{C}_S$ of S-cofibrations by the previous paragraph, since all generators of $\mathbf{C}_S$ are α-bounded. It follows that the lifting problem

$$\begin{tikzcd} A' \arrow[r, "\beta"] \arrow[d, "j"'] & C' \arrow[d, "q"] \\ B' \arrow[r, "1"'] \arrow[ur, dotted] & B' \end{tikzcd}$$

has a solution, so that j is a retract of β.

Chapter 7
Bisimplicial presheaves

7.1 Bisimplicial presheaves

Recall that a *bisimplicial set* X is a functor

$$X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathbf{Set},$$

and a *morphism of bisimplicial sets* is a natural transformation of such functors. Write $X_{p,q} = X(\mathbf{p},\mathbf{q})$ for ordinal numbers $\mathbf{p}$ and $\mathbf{q}$. Let $s^2\mathbf{Set}$ denote the category of bisimplicial sets.

The bisimplicial set $\hom(\ ,(\mathbf{p},\mathbf{q}))$ which is represented by the pair of ordinal numbers $(\mathbf{p},\mathbf{q})$ is denoted by $\Delta^{p,q}$, and is called a *standard bisimplex*. The bisimplices are the cells for the category of bisimplicial sets.

As usual, the *diagonal simplicial set* $d(X)$ is defined by

$$d(X)_p = X_{p,p}.$$

This construction defines the diagonal functor

$$d : s^2\mathbf{Set} \to s\mathbf{Set}.$$

The diagonal functor has both a left adjoint d^* and a right adjoint d_*. The left adjoint d^* is defined by extending the assignment

$$d^*\Delta^n = \Delta^{n,n}$$

in a canonical way, while the right adjoint d_* is defined by

$$d_*(Y)_{p,q} = \hom(\Delta^p \times \Delta^q, Y),$$

All functorial constructions on bisimplicial sets extend to presheaves of bisimplicial sets. Let $\mathscr{C}$ be a small Grothendieck site, and let $s^2\mathbf{Pre}(\mathscr{C})$ denote the category of functors $X : \mathscr{C}^{op} \to s^2\mathbf{Set}$ and all natural transformations between them — this is the category of bisimplicial presheaves, or presheaves of bisimplicial sets on the site $\mathscr{C}$.

Say that a map $f : X \to Y$ of bisimplicial presheaves is a *diagonal weak equivalence* if the induced simplicial presheaf map $d(X) \to d(Y)$ is a local weak equivalence in the usual sense (Definition 4.1). A monomorphism of bisimplicial presheaves is a *cofibration*. An *injective fibration* of bisimplicial presheaves is a morphism which has the right lifting property with respect to trivial cofibrations.

Suppose that β is a cardinal number. A bisimplicial presheaf A is said to be β-*bounded* if $|A_{p,q}(U)| < \beta$ for all $p,q \geq 0$ and all objects U in $\mathscr{C}$.

Suppose that α is an infinite cardinal which is an upper bound for the site $\mathscr{C}$ in the sense that $\alpha > |\mathrm{Mor}(\mathscr{C}|$. We have the following "bounded cofibration lemma" for bisimplicial presheaves:

Lemma 7.1. *Suppose that $i : X \to Y$ is a trivial cofibration of bisimplicial presheaves, and that A is an α-bounded subobject of Y. Then Y has an α-bounded subobject B such that $A \subset B$ and the cofibration $B \cap X \to B$ is a diagonal weak equivalence.*

Proof. There is an induced diagram

$$\begin{tikzcd} & d(X) \arrow[d, "i_*"] \\ d(A) \arrow[r] & d(Y) \end{tikzcd}$$

where i_* is a trivial cofibration of simplicial presheaves and $d(A)$ is an α-bounded subobject of $d(Y)$. The bounded cofibration lemma for simplicial presheaves (this result first appeared as Lemma 12 of [36]) implies that there is an α-bounded subobject D_1 of $d(Y)$ such that $d(A) \subset D_1$ and $D_1 \cap d(X) \to D_1$ is a local weak equivalence. Since D_1 is α-bounded there is an α-bounded subobject A_1 of the bisimplicial presheaf Y such that $A \subset A_1$ and $D_1 \subset d(A_1)$. Repeat this construction inductively to find an ascending families of α-bounded subobjects

$$A \subset A_1 \subset A_2 \subset \cdots \subset Y$$

and

$$d(A) \subset D_1 \subset D_2 \subset \cdots \subset d(Y)$$

such that $D_i \subset d(A_{i+1})$ and the map $D_i \cap d(X) \to D_i$ is a local weak equivalence for all i. Set $B = \cup_i A_i$. Then the map $B \cap X \to B$ of bisimplicial presheaves is a diagonal weak equivalence.

Corollary 7.2. *A map $p : X \to Y$ is an injective fibration of bisimplicial presheaves if and only if it has the right lifting property with respect to all α-bounded trivial cofibrations.*

Proof. The proof is the usual Zorn's lemma argument — see Lemma 5.4.

Recall that every simplicial set K can be identified with a horizontally constant bisimplicial set having the same name in a standard way, with $K_{p,q} = K_q$. One also uses the same notation for a bisimplicial set B and its associated constant simplicial presheaf, so that $B(U) = B$ for all objects U of $\mathscr{C}$.

Lemma 7.3. *A map $q : Z \to Y$ is an injective fibration and a diagonal weak equivalence if and only if it has the right lifting property with respect to all α-bounded cofibrations.*

Proof. If q has the right lifting property with respect to all α-bounded cofibrations, then it has the right lifting property with respect to all cofibrations, by the usual Zorn's lemma argument. In this case, q has a section $\sigma : Y \to Z$, and the lifting exists in the diagram

$$\begin{tikzcd} Z \sqcup Z \arrow[rr, "(\sigma q{,}1)"] \arrow[d] & & Z \arrow[d, "q"] \\ Z \times \Delta^1 \arrow[r, "pr"'] \arrow[urr, dotted] & Z \arrow[r, "q"'] & Y \end{tikzcd}$$

It follows that the induced map $d(q)$ is a simplicial homotopy equivalence, and hence a local weak equivalence.

Suppose that q is an injective fibration and a diagonal weak equivalence. Then q has a factorization

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "q"'] & X \arrow[d, "p"] \\ & Y \end{tikzcd}$$

such that p has the right lifting property with respect to all α-bounded cofibrations and i is a cofibration. Then p is a diagonal weak equivalence, so the cofibration i is a diagonal weak equivalence, and the lift exists in the diagram

$$\begin{tikzcd} Z \arrow[r, "1"] \arrow[d, "i"'] & Z \arrow[d, "q"] \\ X \arrow[r, "p"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

The map q is therefore a retract of the map p, and has the right lifting property with respect to all α-bounded cofibrations.

The function complex $\mathbf{hom}(X,Y)$ for bisimplicial sets X and Y is the simplicial set whose n-simplices are the bisimplicial set maps $X \times \Delta^n \to Y$.

Theorem 7.4. *Suppose that $\mathscr{C}$ is a small Grothendieck site. Then, with the definitions of cofibration, injective fibration and diagonal weak equivalence given above, the category $s^2\mathbf{Pre}(\mathscr{C})$ of bisimplicial sets has the structure of a cofibrantly generated closed simplicial model category.*

Properness for the model structure of Theorem 7.4 is proved in Corollary 7.7 below.

Proof. The axioms **CM1**, **CM2** and **CM3** are easy to verify: in particular, **CM2** and **CM3** are straightforward consequences of the corresponding statements for the injective model structure on simplicial presheaves. Similarly, trivial cofibrations are closed under pushout, so that Corollary 7.2 and Lemma 7.3 imply the factorization axiom **CM5**. The lifting axiom **CM4** also follows from Lemma 7.3. The cofibrant generation follows from Corollary 7.2 and Lemma 7.3.

For the simplicial structure, we show that if $i : A \to B$ is a cofibration of bisimplicial presheaves and $j : K \to L$ is a cofibration of simplicial sets, then the cofibration

$$(B \times K) \cup (A \times L) \to B \times L$$

is trivial if either i or j is trivial, but this is a consequence of the corresponding statement for simplicial presheaves.

Remark 7.5. The model structure of Theorem 7.4 is the *diagonal structure* on the category of bisimplicial presheaves. This result specializes to give diagonal model structures for all categories $s^2\mathbf{Set}^I$ of small diagrams of simplicial sets and to the category $s^2\mathbf{Set}$.

In particular, a cofibration for the diagonal structure on bisimplicial sets is a monomorphism, a weak equivalences is a bisimplicial set map $X \to Y$ such that the induced map $d(X) \to d(Y)$ is a weak equivalence of simplicial sets, and injective fibrations are defined by a right lifting property with respect to trivial cofibrations.

The left adjoint

$$d^* : s\mathbf{Pre}(\mathscr{C}) \to s^2\mathbf{Pre}(\mathscr{C})$$

of the diagonal functor

$$d : s^2\mathbf{Pre}(\mathscr{C}) \to s\mathbf{Pre}(\mathscr{C})$$

therefore preserves cofibrations, and it preserves trivial cofibrations by a Boolean localization argument (Proposition 4.26). It follows that the adjoint functors

$$d^* : s\mathbf{Pre}(\mathscr{C}) \leftrightarrows s^2\mathbf{Pre}(\mathscr{C}) : d$$

define a Quillen adjunction, but we can say more:

Proposition 7.6. *Suppose that $\mathscr{C}$ is a small Grothendieck site. Then the adjoint functors*

$$d^* : s\mathbf{Pre}(\mathscr{C}) \leftrightarrows s^2\mathbf{Pre}(\mathscr{C}) : d$$

define a Quillen equivalence between the injective model structure on simplicial presheaves and the diagonal structure on bisimplicial presheaves on the site $\mathscr{C}$.

Proof. The adjoint functors

$$d^* : s\mathbf{Set} \leftrightarrows s^2\mathbf{Set}$$

define a Quillen equivalence between the standard model structure on simplicial sets and the diagonal structure on bisimplicial sets.

In effect, the adjunction map $\eta : \Delta^n \to dd^*(\Delta^n)$ can be identified up to isomorphism with the diagonal map $\Delta^n \to \Delta^n \times \Delta^n$, which map is a weak equivalence. The functors d and d^* both preserve colimits, cofibrations and trivial cofibrations, so an induction on skeleta shows that the adjunction map $\eta : X \to dd^*(X)$ is a weak equivalence for all simplicial sets X. A triangle identity argument then shows that the natural map $\varepsilon : d^*d(Y) \to Y$ is a diagonal equivalence for all bisimplicial sets Y.

It follows that the adjunction maps $\eta : X \to d^*d(X)$ and $\varepsilon : d^*d(Y) \to Y$ are sectionwise weak equivalences for all bisimplicial presheaves X and simplicial presheaves Y.

Corollary 7.7. *The diagonal model structure on the category $s^2\mathbf{Pre}(\mathscr{C})$ is proper.*

Proof. All bisimplicial presheaves are cofibrant, so that pushouts of diagonal weak equivalences along cofibrations are diagonal weak equivalences [24, II.8.5].

The functor d^* preserves cofibrations and weak equivalences, so that d preserves fibrations. The functor d also preserves pullbacks. Thus, right properness for the diagonal model structure on bisimplicial presheaves follows from right properness for the injective structure on simplicial presheaves.

The Moerdijk model structure is another well known example of a model structure on the category $s^2\mathbf{Set}$ of bisimplicial sets for which the weak equivalences are the diagonal weak equivalences — see [54], and Section IV.3.3 of [24]. The Moerdijk structure is induced from the standard model structure on simplicial sets, in the sense that a bisimplicial set map $X \to Y$ is a fibration (respectively weak equivalence) if and only if the induced map $d(X) \to d(Y)$ is a Kan fibration (respectively weak equivalence) of simplicial sets. The Moerdijk structure is also Quillen equivalent to the standard model structure on simplicial sets, via the diagonal functor and its left adjoint.

Again, let α be an infinite cardinal such that $\alpha > |\mathrm{Mor}(\mathscr{C})|$.

Suppose that S is a set of cofibrations of bisimplicial presheaves which contains the set S_0 of all maps $d^*A \to d^*B$ which are induced by α-bounded cofibrations $A \to B$ of simplicial presheaves. Suppose that S further satisfies the closure property that if the map $C \to D$ is in S, then so is the induced cofibration

$$(D \times \partial\Delta^n) \cup (C \times \Delta^n) \to D \times \Delta^n,$$

for all $n \geq 0$. (Here, $X \times K$, for a bisimplicial set X and a simplicial set K is the product of X with the horizontally constant bisimplicial set associated to K.) Let C_S be the saturation of the set S in the class of all cofibrations (monomorphisms) of the bisimplicial presheaf category. We say that C_S is the class of *S-cofibrations*.

Say that a bisimplicial presheaf map $p : X \to Y$ is an *S-fibration* if it has the right lifting property with respect to all S-cofibrations which are diagonal weak equivalences.

The proof of the following result follows the outline established for Theorem 6.22:

Theorem 7.8. *The category $s^2\mathbf{Pre}(\mathscr{C})$ of bisimplicial presheaves, together with the S-cofibrations, diagonal weak equivalences and S-fibrations satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

Proof. Every map $f : X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where j is a member of C_S and q has the right lifting property with respect to all members of C_S. Then $q_* : d(Z) \to d(Y)$ is a trivial injective fibration of simplicial presheaves, so that q is a diagonal weak equivalence. The map q is an S-fibration.

The map $f : X \to Y$ also has a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & W \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where i is a trivial cofibration and p is a fibration for the diagonal model structure of Theorem 7.4. The map p is an S-fibration. The cofibraton i has a factorization $i = q \cdot j$ as above, where j is an S-cofibration and q is an S-fibration and a diagonal equivalence. The map j is a diagonal equivalence, so that f has a factorization $f = (p \cdot q) \cdot j$ such that $p\dot{q}$ is an S-fibration and j is an S-cofibration and a diagonal equivalence.

We have verified the model category axiom **CM5**. If $p : X \to Y$ is an S-fibration and a diagonal equivalence, then it is a retract of a map which has the right lifting property with respect to all S-cofibrations, giving **CM4**. The rest of the model category axioms are trivial.

The simplicial model axiom **SM7** is a consequence of the construction of the class C_S and the instance of this axiom for the injective model structure on simplicial presheaves. The left properness of this structure is an easy consequence of left properness for the diagonal structure on $s^2\mathbf{Pre}(\mathscr{C})$, while right properness follows from right properness for the injective structure on $s\mathbf{Pre}(\mathscr{C})$.

The cofibrant generation is proved with the usual trick. Every α-bounded trivial cofibration $\beta : A \to B$ has a factorization

$$\begin{tikzcd} A \arrow[r, "j_\beta"] \arrow[dr, "\beta"'] & Z_\beta \arrow[d, "q_\beta"] \\ & B \end{tikzcd}$$

as in the first paragraph, where j_β is an S-cofibration and q_β has the right lifting property with respect to all S-cofibrations. Then both j_β and q_β are diagonal equivalences. One shows that if $i : C \to D$ is an α-bounded S-cofibration and there is a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ D \arrow[r] & Y \end{tikzcd}$$

where f is a diagonal equivalence, then the diagram has a factorization

$$\begin{tikzcd} C \arrow[r] \arrow[d, "i"'] & A \arrow[r] \arrow[d, "j_\beta"] & X \arrow[d, "f"] \\ D \arrow[r] & Z_\beta \arrow[r] & Y \end{tikzcd}$$

for some β.

Finally, if $j : E \to F$ is an S-cofibration and a diagonal equivalence, then j has a factorization

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[dr, "j"'] & V \arrow[d, "p"] \\ & F \end{tikzcd}$$

where p has the right lifting property with respect to all j_β and i is in the saturation of the set of all maps j_β. But then j and p are diagonal equivalences, and the construction of the last paragraph shows that p has the right lifting property with respect to all members of C_S, so that i is a retract of j. This means that the set of all maps j_β generates the class of trivial cofibrations in the model structure defined by the set of cofibrations S.

Say that the model structure of Theorem 7.8 is the *S-model structure* on the category of bisimplicial presheaves.

The S_0-model structure on bisimplicial sets (for whatever infinite cardinal α) is the Moerdijk structure, and the S_0-model structure for bisimplicial presheaves is a locally defined analogue of the Moerdijk structure. An obvious comparison with the various intermediate model structures for simplicial presheaves of Theorem 6.22 says that the S_0-model structure for bisimplicial presheaves is a "projective" model structure, while the diagonal model structure of Theorem 7.4 is an "injective" model structure, and all S-model structures have classes of cofibrations lying between these two extremes.

7.2 Bisimplicial sets

Suppose that K and L are simplicial sets, and let $K\tilde{\times}L$ be the bisimplical set defined by

$$(K\tilde{\times}L)_{p,q} = K_p \times L_q.$$

The bisimplicial set $K\tilde{\times}L$ is the *external product* of K and L.

Examples: 1) The standard bisimplex $\Delta^{p,q}$ has the form

$$\Delta^{p,q} = \Delta^p \tilde{\times} \Delta^q.$$

2) Set

$$\partial\Delta^{p,q} = (\partial\Delta^p \tilde{\times} \Delta^q) \cup (\Delta^p \tilde{\times} \partial\Delta^q) \subset \Delta^p \tilde{\times} \Delta^q = \Delta^{p,q}.$$

Then the *boundary* $\partial\Delta^{p,q}$ of the bisimplex $\Delta^{p,q}$ is generated as a subcomplex by the images of the maps $(d^i,1) : \Delta^{p-1,q} \to \Delta^{p,q}$ and $(1,d^j) : \Delta^{p,q-1} \to \Delta^{p,q}$.

The following statement about simplicial sets is well known — it is sometimes called the Eilenberg-Zilber Lemma (see [15, (8.3)]) and is used, however silently [24, I.2.3], in discussions of the standard skeletal decomposition of a simplicial set. The proof is usually left as an exercise.

Lemma 7.9. *Suppose that x,y are non-degenerate simplices of a simplicial set X, and suppose that s,t are ordinal number epimorphisms such that $s^*(x) = t^*(y)$. Then $x = y$ and $s = t$.*

Suppose that X is a bisimplicial set and that $x \in X_{p,q}$. The number $p+q$ is the *total degree* of x.

Suppose that A is a subcomplex of a bisimplicial set X and that $x \in X_{p,q}$ is a bisimplex of $X - A$ of minimal total degree. Write $x : \Delta^{p,q} \to X$ for the classifying map of the bisimplex x. The bisimplices $(d_i,1)(x)$ and $(1,d_j)(x)$ have smaller total degree than x and are therefore in A, and it follows that there is a pullback diagram

$$\begin{tikzcd}
\partial\Delta^{p,q} \arrow[r, "\alpha"] \arrow[d] & A \arrow[d, "i"] \\
\Delta^{p,q} \arrow[r, "x"'] & X
\end{tikzcd}$$

of bisimplicial set maps.

Lemma 7.10. *Suppose that A is a subcomplex of a bisimplicial set X and that $x \in X_{p,q}$ is a bisimplex of $X - A$ of minimal total degree. Form the pushout*

$$\begin{tikzcd}
\partial\Delta^{p,q} \arrow[r, "\alpha"] \arrow[d] & A \arrow[d, "i"] \\
\Delta^{p,q} \arrow[r, "x"'] & B
\end{tikzcd}$$

Then the induced bisimplicial set map $B \to X$ is a monomorphism.

Proof. If $x = s(y)$ for some degeneracy s (vertical or horizontal), then y has smaller total degree, and so $y \in A$ and $x \in A$. It follows that x is vertically and horizontally non-degenerate.

There is a decomposition

$$B_{r,s} = A_{r,s} \sqcup \{u \times v : \mathbf{r} \times \mathbf{s} \to \mathbf{p} \times \mathbf{q},\ u, v \text{ epi}\}.$$

in all bidegrees.

If $a \in A_{r,s}$ and $u \times v$ have the same image in X, then $a = (u \times v)^*(x)$ is in A so that $x \in A$ by applying a suitable section of $u \times v$, which can't happen. The restriction of $B_{r,s} \to X_{r,s}$ to $A_{r,s}$ is the monomorphism $i : A_{r,s} \to X_{r,s}$. Finally, if the epis $u \times v, u' \times v' : \mathbf{r} \times \mathbf{s} \to \mathbf{p} \times \mathbf{q}$ have the same image in X, then $(u \times v)^*(x) = (u' \times v')^*(x)$ in X.

The bisimplex $(1 \times v)^*(x)$ is horizontally non-degenerate. Otherwise,

$$(1 \times v)^*(x) = (s \times 1)^*(y)$$

for some y and non-trivial ordinal number epi s, and if d is a section of v then

$$x = (1 \times d)^*(1 \times v)^*(x) = (1 \times d)^*(s \times 1)^*(y) = (s \times 1)^*(1 \times d)^*(y)$$

so that x is horizontally degenerate. Similarly, $(1 \times v')^*(x)$ is horizontally non-degenerate, and so Lemma 7.9 and the relations

$$(u \times 1)^*(1 \times v)^*(x) = (u' \times 1)^*(1 \times v')^*(x)$$

together imply that $u = u'$ and $(1 \times v)^*(x) = (1 \times v')^*(x)$, so that $v = v'$

Corollary 7.11. *The set of inclusions $\partial\Delta^{p,q} \subset \Delta^{p,q}$ generates the class of cofibrations of s^2*$\mathbf{Set}$.

The class $\mathscr{A}$ of *anodyne extensions* of $s^2\mathbf{Set}$ is the saturation of the set of bisimplicial set maps S, which consists of all morphisms

$$(\Lambda^r_k \tilde{\times} \Delta^s) \cup (\Delta^r \tilde{\times} \partial\Delta^s) \subset \Delta^r \tilde{\times} \Delta^s = \Delta^{r,s}$$

as well as all morphisms

$$(\partial\Delta^r \tilde{\times} \Delta^s) \cup (\Delta^r \tilde{\times} \Lambda^s_j) \subset \Delta^r \tilde{\times} \Delta^s = \Delta^{r,s}$$

The class $\mathscr{A}$ contains the set of all cofibrations

$$(A \tilde{\times} D) \cup (B \tilde{\times} C) \subset B \tilde{\times} D$$

induced by cofibrations $A \to B$ and $C \to D$, where one of the two maps is a trivial cofibration of simplicial sets. The diagonal of such a map is the trivial cofibration

$$(A \times D) \cup (B \times C) \subset B \times D.$$

in simplicial sets.

In particular, we have the following:

Lemma 7.12. *Every anodyne extension of bisimplicial sets is a diagonal weak equivalence.*

Say that a map $p : X \to Y$ of bisimplicial sets is a *Kan fibration* if it has the right lifting property with respect to all anodyne extensions. Every injective fibration is a Kan fibration.

The purpose of this section is to prove the converse assertion, so that the injective fibrations of bisimplicial sets are precisely the Kan fibrations. This statement appears as Theorem 7.22 below.

Suppose that X is a bisimplicial set and that K is a simplicial set. Recall that the bisimplicial set $X \times K$ has

$$(X \times K)_{p,q} = X_{p,q} \times K_q.$$

There is a natural isomorphism

$$d(X \times K) \cong d(X) \times K.$$

The construction $(X,K) \mapsto X \times K$ preserves diagonal weak equivalences in bisimplicial sets X and weak equivalences in simplicial sets K.

Lemma 7.13. *Suppose that $i : A \to B$ is a cofibration of bisimplicial sets and that $j : K \to L$ is a cofibration of simplicial sets. Then the induced map*

$$(i, j)_* : (B \times K) \cup (A \times L) \to B \times L$$

is a cofibration which is an anodyne extension if either i or j is an anodyne extension.

Recall that an anodyne extension of simplicial sets is a trivial cofibration.

Proof. The map

$$(\Delta^{r,s} \times K) \cup (\partial\Delta^{r,s} \times L) \to \Delta^{r,s} \times L$$

can be identified with the map

$$(\partial\Delta^r \tilde{\times} (\Delta^s \times L)) \cup (\Delta^r \tilde{\times} ((\partial\Delta^s \times L) \cup (\Delta^s \times K))) \to \Delta^r \tilde{\times} (\Delta^s \times L),$$

which is a cofibration.

The simplicial set map

$$(\partial\Delta^s \times L) \cup (\Delta^s \times K) \to \Delta^s \times L$$

is an anodyne extension if j is anodyne, so that $(i, j)_*$ is an anodyne extension in general if j is anodyne.

The remaining statements have similar proofs.

Suppose that X and Y are bisimplicial sets. The collection of bisimplicial set maps

$$X \times \Delta^n \to Y$$

is the set of n-simplices of the simplicial set $\mathbf{hom}(X,Y)$. If $p : X \to Y$ is a Kan fibration and A is a bisimplicial set, then he induced map $p_* : \mathbf{hom}(A,X) \to \mathbf{hom}(A,Y)$ is a fibration of simplicial sets since all maps $A \times \Lambda^n_k \to A \times \Delta^n$ are anodyne extensions by Lemma 7.13.

If $f : A \to Y$ is a map of bisimplicial sets, then f is a vertex of the simplicial set $\mathbf{hom}(A,Y)$, and we can form the pullback diagram

$$\begin{tikzcd} \mathbf{hom}_Y(A,X) \arrow[r] \arrow[d] & \mathbf{hom}(A,X) \arrow[d, "p_*"] \\ * \arrow[r, "f"'] & \mathbf{hom}(A,Y) \end{tikzcd}$$

The simplicial set $\mathbf{hom}_Y(A,X)$ is a Kan complex since p is a Kan fibration.

The n-simplices of $\mathbf{hom}_Y(A,X)$ are commutative diagrams of the form

$$\begin{tikzcd} A \times \Delta^n \arrow[r] \arrow[d, "pr"'] & X \arrow[d, "p"] \\ A \arrow[r, "f"'] & Y \end{tikzcd}$$

It follows that the functor $s\mathbf{Set}/Y \to s\mathbf{Set}$ which takes an object $f : A \to Y$ to the simplicial set $\mathbf{hom}_Y(A,X)$ has a left adjoint which takes a simplicial set K to the object

$$A \times K \xrightarrow{pr} A \xrightarrow{f} Y.$$

A map

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr] & & B \arrow[dl] \\ & Y & \end{tikzcd}$$

of bisimplicial sets over Y is said to be an *anodyne equivalence over Y* if the simplicial set maps

$$\mathbf{hom}_Y(B,X) \xrightarrow{\alpha^*} \mathbf{hom}_Y(A,X)$$

are weak equivalences for all Kan fibrations $p : X \to Y$.

Lemma 7.14. *Suppose that the map $A \xrightarrow{\alpha} B \to Y$ of bisimplicial sets over Y is defined by a cofibration α, and suppose that $p : X \to Y$ is a Kan fibration. Then the induced map*

$$\alpha^* : \mathbf{hom}_Y(B,X) \to \mathbf{hom}_Y(A,X)$$

is a Kan fibration. If α is an anodyne extension, then α^* is a trivial Kan fibration.

Proof. Use Lemma 7.13 to see that the lifting exists in all diagrams

$$\begin{tikzcd}
(B \times \Lambda^n_k) \cup (A \times \Delta^n) \arrow[rr] \arrow[d] & & X \arrow[d, "p"] \\
B \times \Delta^n \arrow[r, "pr"'] \arrow[urr, dotted] & B \arrow[r, "g"'] & Y
\end{tikzcd}$$

Similarly, if $\alpha : A \to B$ is an anodyne extension, then the lifting exists in all diagrams

$$\begin{tikzcd}
(B \times \partial\Delta^n) \cup (A \times \Delta^n) \arrow[rr] \arrow[d] & & X \arrow[d, "p"] \\
B \times \Delta^n \arrow[r, "pr"'] \arrow[urr, dotted] & B \arrow[r, "g"'] & Y
\end{tikzcd}$$

so that α^* is a trivial fibration.

Corollary 7.15. *Suppose that $\alpha : A \to B$ is an anodyne extension. Then any map $A \xrightarrow{\alpha} B \to Y$ of bisimplicial sets over Y is an anodyne equivalence.*

Lemma 7.16. *If $f : K \to K'$ and $g : L \to L'$ are weak equivalences of simplicial sets, then any map*

$$f \tilde{\times} g : K \tilde{\times} L \to K' \tilde{\times} L' \to Y$$

is an anodyne equivalence of bisimplicial sets over Y.

Proof. We show that the map

$$f \times 1 : K \tilde{\times} L \to K' \tilde{\times} L \to Y$$

is an anodyne weak equivalence.

This is true if f is a trivial cofibration by Corollary 7.15, and is therefore true in general since all simplicial sets are cofibrant.

If X is a bisimplicial set, then the simplicial set maps

$$\Delta^n \times X_{n,m} \to X_{*,m}$$

induce bisimplicial set maps

$$\gamma_n : \Delta^n \tilde{\times} X_n \to X.$$

The bisimplicial set X has a natural filtration $\mathrm{sk}_n X$ by (horizontal) skeleta, and there are natural pushout diagrams

$$\begin{tikzcd} s_{[r]}X_{n-1} \arrow[r, "s_{r+1}"] \arrow[d] & s_{[r]}X_n \arrow[d] \\ X_n \arrow[r, "s_{r+1}"'] & s_{[r+1]}X_n \end{tikzcd} \tag{7.1}$$

and

$$\begin{tikzcd} (\Delta^{n+1} \tilde{\times} s_{[n]}X_n) \cup (\partial\Delta^{n+1} \tilde{\times} X_{n+1}) \arrow[r] \arrow[d] & \mathrm{sk}_n X \arrow[d] \\ \Delta^{n+1} \tilde{\times} X_{n+1} \arrow[r] & \mathrm{sk}_{n+1} X \end{tikzcd} \tag{7.2}$$

in bisimplicial sets, in which the vertical maps are cofibrations.

Lemma 7.17. *Suppose that $A \xrightarrow{f} B \to Y$ is a map of bisimplicial sets over Y such that the map $f : A_n \to B_n$ is a weak equivalence of simplicial sets in each horizontal degree n. Then the map f is an anodyne weak equivalence over Y.*

Proof. Suppose that $p : X \to Y$ is a Kan fibration.

The functor which takes $A \to Y$ to $\mathbf{hom}_Y(A,X)$ has a left adjoint, and takes cofibrations to Kan fibrations by Lemma 7.14. It follows that anodyne weak equivalences satisfy a patching property for pushouts along cofibrations. One can then show inductively that the maps $s_{[r]}A \to s_{[r]}B \to Y$ and $\mathrm{sk}_n A \to \mathrm{sk}_n B \to Y$ are anodyne equivalences over Y.

The vertical maps in the diagram

$$\begin{tikzcd} \mathrm{sk}_0 A \arrow[r] \arrow[d] & \mathrm{sk}_1 A \arrow[r] \arrow[d] & \mathrm{sk}_2 A \arrow[r] \arrow[d] & \cdots \\ \mathrm{sk}_0 B \arrow[r] & \mathrm{sk}_1 B \arrow[r] & \mathrm{sk}_2 B \arrow[r] & \cdots \end{tikzcd}$$

are anodyne weak equivalences, and the horizontal maps are cofibrations. It follows that the induced map

$$\mathbf{hom}_Y(B,X) \cong \varprojlim_n \mathbf{hom}_Y(\mathrm{sk}_n B, X) \to \varprojlim_n \mathbf{hom}_Y(\mathrm{sk}_n A, X) = \mathbf{hom}_Y(A,X)$$

is a weak equivalence.

Lemma 7.18. *Suppose that the map*

$$\begin{tikzcd} Z \arrow[rr, "\pi"] \arrow[dr, "f"'] & & W \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

of bisimplicial sets over Y is defined by a trivial fibration $\pi : Z \to W$. Then the map $\pi : f \to g$ is an anodyne equivalence.

Proof. The composite

$$Z \times \Delta^1 \xrightarrow{pr} Z \xrightarrow{f} X$$

is a cylinder for f in $s^2\mathbf{Set}/Y$.

The map π is a trivial fibration of $s^2\mathbf{Set}/Y$, and all objects of this category are cofibrant. It follows that the map $\pi : f \to g$ is a fibre homotopy equivalence, in the sense that one uses the cylinder above. If the maps $\alpha, \beta : Z \to W \to X$ are fibre homotopic, then they induce the same maps

$$\alpha^*, \beta^* : \mathbf{hom}_Y(W,X) \to \mathbf{hom}_Y(Z,X)$$

in the homotopy category for simplicial sets, for all Kan fibrations $p : X \to Y$. It follows that the map

$$\pi^* : \mathbf{hom}_Y(W,X) \to \mathbf{hom}_Y(Z,X)$$

induces an isomorphism in the homotopy category, and is therefore a weak equivalence of simplicial sets for all Kan fibrations $p : X \to Y$.

Lemma 7.19. *Suppose that every weak equivalence $\alpha : f \to g$ over a bisimplicial set Y is an anodyne equivalence. Then every Kan fibration $p : X \to Y$ is a diagonal fibration.*

Proof. Suppose we have a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where i is a trivial cofibration and p is a Kan fibration. There is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[d, "i"'] & LA \arrow[d, "i_*"] \arrow[dr, "p_A"] & \\ B \arrow[r, "j_B"'] & LB \arrow[r, "p_B"'] & Y \end{tikzcd}$$

such that j_A and j_B are anodyne extensions and p_A and p_B are Kan fibrations. There is a lifting α making the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j_A"'] & X \arrow[d, "p"] \\ LA \arrow[r, "p_A"'] \arrow[ur, "\alpha"] & Y \end{tikzcd}$$

since p is a Kan fibration and j_A is an anodyne extension. Find a factorization

$$\begin{tikzcd} LA \arrow[rr, "i_*"] \arrow[dr, "j"'] & & LB \\ & Z \arrow[ur, "\pi"'] & \end{tikzcd}$$

such that π is a trivial fibration and j is a cofibration. Then π is an anodyne equivalence (Lemma 7.18), so that the cofibration j is an anodyne equivalence.

Now consider the lifting problem

$$\begin{tikzcd} LA \arrow[r, "\alpha"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ Z \arrow[r, "p_B\pi"'] \arrow[ur, dotted, "\gamma"] & Y \end{tikzcd}$$

Generally, every Kan fibration $p : X \to Y$ has the right lifting property with respect to all cofibrations $j : C \to D$ which are anodyne equivalences, since the corresponding maps $\mathbf{hom}_Y(D,X) \to \mathbf{hom}_Y(C,X)$ are trivial fibrations and are therefore surjective in degree 0. It follows that the indicated lifting γ exists.

Finally, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[dd, "i"'] & LA \arrow[r, "\alpha"] \arrow[d, "j"] & X \arrow[dd, "p"] \\ & Z \arrow[ur, "\gamma"] \arrow[d, "\pi"] & \\ B \arrow[r, "j_B"'] \arrow[ur, dotted] & LB \arrow[r, "p_B"'] & Y \end{tikzcd}$$

where the indicated lift exists since i is a cofibration and π is a trivial fibration.

Lemma 7.20. *Suppose that in the diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi'"] \\ & \Delta^{p,q} & \end{tikzcd}$$

the map f is a diagonal weak equivalence. Then this diagram is an anodyne equivalence over $\Delta^{p,q}$.

Proof. We can suppose that the maps π and π' are Kan fibrations.

If $\pi : X \to \Delta^{p,q}$ is a Kan fibration, then all maps

$$X_n \to \bigsqcup_{\mathbf{n} \to \mathbf{p}} \Delta^q$$

are fibrations of simplicial sets, and all diagrams

$$\begin{tikzcd}
X_n \arrow[r, "\theta^*"] \arrow[d] & X_m \arrow[d] \\
\bigsqcup_{\mathbf{n}\to\mathbf{p}} \Delta^q \arrow[r, "\theta^*"'] & \bigsqcup_{\mathbf{m}\to\mathbf{p}} \Delta^q
\end{tikzcd}$$

are homotopy cartesian. In particular, the bisimplicial set X is determined by simplicial sets X_σ, one for each $\sigma : \mathbf{n} \to \mathbf{p}$, and weak equivalences $X_\sigma \to X_{\sigma\theta}$ which are functorial in maps between simplices of Δ^p. Write $h(X)$ for the homotopy colimit of the diagram $\sigma \mapsto X_\sigma$: it is convenient in this case to write $h(X) = \varinjlim U_\sigma$ where $U_\sigma \to X_\sigma$ is a projective cofibrant model for the diagram $\sigma \mapsto X_\sigma$. All maps $X_\sigma \to h(X)$ are weak equivalences by Quillen's Theorem B, and so the bisimplicial set map $\pi : X \to \Delta^{p,q}$ is levelwise equivalent (hence anodyne equivalent, by Lemma 7.17) to a map $\Delta^p \tilde{\times} h(X) \to \Delta^{p,q}$. This identification is natural in Kan fibrations π, so the commutative diagram

$$\begin{tikzcd}
\Delta^p \tilde{\times} h(X) \arrow[rr, "1\tilde{\times} f_*"] \arrow[dr] & & \Delta^p \tilde{\times} h(Y) \arrow[dl] \\
& \Delta^p \tilde{\times} \Delta^q &
\end{tikzcd}$$

is anodyne equivalent to the original. The map $1\tilde{\times} f_*$ is a diagonal equivalence, so that the simplicial set map $f_* : h(X) \to h(Y)$ is a weak equivalence. It follows from Lemma 7.16 that the map $1\tilde{\times} f_*$ is an anodyne equivalence, as is the map f.

The following result is then a consequence of Lemma 7.19 and Lemma 7.20:

Corollary 7.21. *Every Kan fibration $p : X \to \Delta^{p,q}$ is a diagonal fibration.*

Theorem 7.22. *The map $p : X \to Y$ is a diagonal fibration if and only if it is a Kan fibration.*

Proof. We show that every Kan fibration which is a diagonal weak equivalence has the right lifting property with respect to all cofibrations.

Suppose that this is so, and let $i : A \to B$ be a cofibration which is a diagonal weak equivalence. Find a factorization

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[dr, "i"'] & Z \arrow[d, "p"] \\
& B
\end{tikzcd}$$

such that j is anodyne and p is a Kan fibration. Then, subject to the claim of the first paragraph, the map p is a diagonal weak equivalence and the lifting exists in the diagram

$$
\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "i"'] & Z \arrow[d, "p"] \\
B \arrow[r, "1"'] \arrow[ur, dotted] & B
\end{tikzcd}
$$

Then i is a retract of j, and is therefore an anodyne extension. Thus, the classes of diagonal trivial cofibrations and anodyne extensions coincide, so the classes of diagonal fibrations and Kan fibrations coincide.

Suppose that $p : X \to Y$ is a Kan fibration and a diagonal equivalence. Form the pullback diagrams

$$
\begin{tikzcd}
p^{-1}(\sigma) \arrow[r] \arrow[d, "p_*"'] & X \arrow[d, "p"] \\
\Delta^{p,q} \arrow[r, "\sigma"'] & Y
\end{tikzcd}
\tag{7.3}
$$

for all bisimplices σ. If

$$
\begin{tikzcd}
\Delta^{r,s} \arrow[rr] \arrow[dr, "\tau"'] & & \Delta^{p,q} \arrow[dl, "\sigma"] \\
& Y &
\end{tikzcd}
$$

is a map of simplices, then the maps p_* in the pullback diagram

$$
\begin{tikzcd}
p^{-1}(\tau) \arrow[r] \arrow[d, "p_*"'] & p^{-1}(\sigma) \arrow[d, "p_*"] \\
\Delta^{r,s} \arrow[r] & \Delta^{p,q}
\end{tikzcd}
$$

are diagonal fibrations by Corollary 7.21, so that the map $p^{-1}(\tau) \to p^{-1}(\sigma)$ is a diagonal equivalence. It follows from Quillen's Theorem B that all diagrams (7.3) are homotopy cartesian for the diagonal model structure.

It follows in particular that the maps p_* are diagqonal equivalences, so that the lifts exist in all diagrams

$$
\begin{tikzcd}
\partial\Delta^{p,q} \arrow[r] \arrow[d] & p^{-1}(\sigma) \arrow[d, "p_*"] \\
\Delta^{p,q} \arrow[r, "1"'] \arrow[ur, dotted] & \Delta^{p,q}
\end{tikzcd}
$$

The map $p : X \to Y$ is therefore a trivial diagonal fibration.

Part III
Sheaf cohomology theory

Chapter 8
Homology sheaves and cohomology groups

8.1 Chain complexes

Suppose that $\mathscr{C}$ is a fixed Grothendieck site, and suppose that R is a presheaf of commutative rings with unit on $\mathscr{C}$.

Write $\mathbf{Pre}_R = \mathbf{Pre}_R(\mathscr{C})$ for the category of R-modules, or abelian presheaves which have an R-module structure. Then $s\mathbf{Pre}_R$ is the category of simplicial R-modules, $\mathrm{Ch}_+(\mathbf{Pre}_R)$ is the category of positively graded (ie. ordinary) chain complexes in $\mathbf{Pre}_R$, and $\mathrm{Ch}(\mathbf{Pre}_R)$ is the category of unbounded chain complexes in $\mathbf{Pre}_R$.

Much of the time in applications, R is a constant presheaf of rings such as $\mathbb{Z}$ or $\mathbb{Z}/n$. In particular, $\mathbf{Pre}_{\mathbb{Z}}$ is the category of presheaves of abelian groups, $s\mathbf{Pre}_{\mathbb{Z}}$ is presheaves of simplicial abelian groups, and $\mathrm{Ch}(\mathbb{Z})$ and $\mathrm{Ch}_+(\mathbb{Z})$ are categories of presheaves of chain complexes. The category $\mathbf{Pre}_{\mathbb{Z}/n}$ is the category of n-torsion abelian presheaves, and so on.

All of these categories have corresponding sheaf categories, based on the category $\mathbf{Shv}_R$ of sheaves of R-modules. Thus, $s\mathbf{Shv}_R$ is the category of simplicial sheaves in R-modules, $\mathrm{Ch}_+(\mathbf{Shv}_R)$ is the category of positively graded chain complexes in $\mathbf{Shv}_R$, and $\mathrm{Ch}(\mathbf{Shv}_R)$ is the category of unbounded complexes.

There is a free R-module functor

$$R : s\mathbf{Pre}(\mathscr{C}) \to s\mathbf{Pre}_R,$$

written $X \mapsto R(X)$ for simplicial presheaves X, where $R(X)_n$ is the free R-module on the presheaf X_n. This functor is left adjoint to the obvious forgetful functor

$$u : s\mathbf{Pre}_R \to s\mathbf{Pre}(\mathscr{C}).$$

The sheaf associated to $R(X)$ is denoted by $\tilde{R}(X)$.

I shall also write $R(X)$ for the associated (presheaf of) Moore chains on X, which is the complex with $R(X)_n$ in degree n and boundary maps

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : R(X)_n \to R(X)_{n-1}.$$

The *homology sheaf* $\tilde{H}_n(X,R)$ is the sheaf associated to the presheaf $H_n(R(X))$. More generally, if A is an R-module, then $\tilde{H}_n(X,A)$ is the sheaf associated to the presheaf $H_n(R(X) \otimes A)$.

The normalized chains functor induces a functor

$$N : s\mathbf{Pre}_R \to \mathrm{Ch}_+(\mathbf{Pre}_R),$$

which is part of an equivalence of categories (the Dold-Kan correspondence [24, III.2.3])

$$N : s\mathbf{Pre}_R \simeq \mathrm{Ch}_+(\mathbf{Pre}_R) : \Gamma.$$

The normalized chain complex NA is the complex with

$$NA_n = \cap_{i=0}^{n-1} \ker(d_i)$$

and boundary

$$\partial = (-1)^n d_n : NA_n \to NA_{n-1}.$$

It is well known [24, III.2.4] that the obvious natural inclusion $NA \subset A$ of NA in the Moore chains is split by collapsing by degeneracies, and this map induces a natural isomorphism

$$H_*(NA) \cong H_*(A)$$

of homology presheaves, and hence an isomorphism

$$\tilde{H}_*(NA) \cong \tilde{H}_*(A)$$

of homology sheaves.

Lemma 8.1. *Suppose that $f : X \to Y$ is a local weak equivalence of simplicial presheaves. Then the induced map $f_* : R(X) \to R(Y)$ of simplicial R-modules is also a local weak equivalence.*

Proof. It is enough to show that if $f : X \to Y$ is a local equivalence of locally fibrant simplicial sheaves, then $f_* : \tilde{R}(X) \to \tilde{R}(Y)$ is a local equivalence of simplicial abelian sheaves, where R is a sheaf of rings.

It suffices to assume that the map $f : X \to Y$ is a morphism of locally fibrant simplicial sheaves on a complete Boolean algebra $\mathscr{B}$, since the inverse image functor p^* for a Boolean localization $p : \mathbf{Shv}(\mathscr{B}) \to \mathbf{Shv}(\mathscr{C})$ commutes with the free R-module functor.

In this case, the map $f : X \to Y$ is a sectionwise weak equivalence, so that $f_* : R(X) \to R(Y)$ is a sectionwise weak equivalence, and so the map $f_* : \tilde{R}(X) \to \tilde{R}(Y)$ of associated sheaves is a local weak equivalence.

Remark 8.2. At one time, Lemma 8.1 and its variants were forms of the *Illusie conjecture* . There are various proofs of this result in the literature: the earliest, by van Osdol [64], is one of the first applications of Boolean localization. See also [31].

Suppose that A is a simplicial abelian group. Then A is a Kan complex, and we know [24, III.2.7] that there is a natural isomorphism

$$\pi_n(A,0) \cong H_n(NA)$$

for $n \geq 0$. There is a canonical isomorphism

$$\pi_n(A,0) \xrightarrow{\cong} \pi_n(A,a)$$

which is defined for any $a \in A_0$ by $[\alpha] \mapsto [\alpha + a]$ where we have written a for the composite

$$\Delta^n \to \Delta^0 \xrightarrow{a} A$$

The collection of these isomorphisms, taken together, define isomorphisms

$$\begin{tikzcd} \pi_n(A,0)\times A_0 \arrow[rr, "\cong"] \arrow[dr, "pr"'] & & \pi_n A \arrow[dl] \\ & A_0 & \end{tikzcd}$$

of abelian groups fibred over A_0, and these isomorphisms are natural in simplicial abelian group homomorphisms.

Lemma 8.3. *A map $A \to B$ of simplicial R-modules induces a local weak equivalence $u(A) \to u(B)$ of simplicial presheaves if and only if the induced map $NA \to NB$ induces an isomorphism in all homology sheaves.*

Proof. If $NA \to NB$ induces an isomorphism in all homology sheaves, then the map $\tilde{\pi}_0(A) \to \tilde{\pi}_0(B)$ and all maps $\tilde{\pi}_n(A,0) \to \tilde{\pi}_n(B,0)$ are isomorphisms of sheaves. The diagram of sheaves associated to the presheaf diagram

$$\begin{tikzcd} \pi_n(A,0)\times A_0 \arrow[r] \arrow[d] & \pi_n(B,0)\times B_0 \arrow[d] \\ A_0 \arrow[r] & B_0 \end{tikzcd}$$

is a pullback if and only if the map $\tilde{\pi}_n(A,0) \to \tilde{\pi}_n(B,0)$ is an isomorphism of sheaves.

Corollary 8.4. *Suppose given a pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "i_*"] \\ B \arrow[r] & D \end{tikzcd}$$

in simplicial R-modules, such that the map i is a monomorphism and a homology sheaf isomorphism. Then the induced map i_* is a homology sheaf isomorphism.

Proof. The cokernel of the monomorphism i_* is B/A, which is acyclic in the sense that $\tilde{H}_*(B/A) = 0$. The Moore chains functor is exact, and the short exact sequence

$$0 \to C \xrightarrow{i_*} D \to B/A \to 0$$

of simplicial R-modules induces a long exact sequence

$$\begin{aligned} \dots \to \tilde{H}_n(C) \xrightarrow{i_*} \tilde{H}_n(D) \to \tilde{H}_n(B/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \dots \\ \xrightarrow{\partial} \tilde{H}_0(C) \xrightarrow{i_*} \tilde{H}_0(D) \to \tilde{H}_0(B/A) \to 0 \end{aligned}$$

in homology sheaves. It follows that all maps

$$\tilde{H}_n(C) \xrightarrow{i_*} \tilde{H}_n(D)$$

are isomorphisms.

Say that a map $f : A \to B$ of simplicial R-modules is a *local weak equivalence* (respectively *injective fibration*) if the simplicial presheaf map $u(A) \to u(B)$ is a local weak equivalence (respectively injective fibration).

A *cofibration* of simplicial R-modules is a map which has the left lifting property with respect to all trivial injective fibrations.

In view of Lemma 8.3, $f : A \to B$ is a local weak equivalence if and only if the induced maps $NA \to NB$ and $A \to B$ of normalized and Moore chains, respectively, are homology sheaf isomorphisms. Homology sheaf isomorphisms are often called *quasi-isomorphisms* .

If $i : A \to B$ is a cofibration of simplicial presheaves, then the induced map $i_* : R(A) \to R(B)$ is a cofibration of simplicial R-modules. The map i_* is a monomorphism, because the free R-module functor preserves monomorphisms.

Analogous definitions are available for maps of simplicial sheaves of R-modules. Say that a map $f : A \to B$ in $s\mathbf{Shv}_R$ is a *local weak equivalence* (respectively *injective fibration*) if the underlying simplicial sheaf map $u(A) \to u(B)$ is a local weak equivalence (respectively injective fibration). Cofibrations are defined by a left lifting property with respect to trivial fibrations.

If $i : A \to B$ is a cofibration of simplicial presheaves, then the induced map $i_* : \tilde{R}(A) \to \tilde{R}(B)$ is a cofibration of $s\mathbf{Shv}_R$. This induced map is also a monomorphism.

Theorem 8.5. *1) With these definitions, the category $s\mathbf{Pre}_R$ of simplicial R-modules satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

2) *With these definitions, the category $s\mathbf{Shv}_R$ of simplicial sheaves of R-modules satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*
3) *The inclusion and associated sheaf functors define a Quillen equivalence*

$$L^2 : s\mathbf{Pre}_R \leftrightarrows s\mathbf{Shv}_R : i$$

between the (injective) model structures of parts 1) and 2).

Proof. The injective model structure on $s\mathbf{Pre}$ is cofibrantly generated. It follows from this, together with Lemma 8.1 and Corollary 8.4, that every map $f : A \to B$ of $s\mathbf{Pre}_R$ has factorizations

$$\begin{tikzcd} & C \arrow[dr, "p"] & \\ A \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "j"'] & & B \\ & D \arrow[ur, "q"'] & \end{tikzcd}$$

such that p is an injective fibration, i is a trivial cofibration which has the left lifting property with respect to all fibrations, q is a trivial injective fibration, j is a cofibration, and both i and j are monomorphisms. This proves the factorization axiom **CM5**. It follows as well that every trivial cofibration is a retract of a map of the form i and therefore has the left lifting property with respect to all fibrations, giving **CM4**. The remaining closed model axioms for the category $s\mathbf{Pre}_R$ of simplicial R-modules are easy to verify.

The generating set $A \to B$ of cofibrations (respectively trivial cofibrations) for simplicial presheaves induces a generating set $R(A) \to R(B)$ of cofibrations (respectively trivial cofibrations) for the category of simplicial R-modules.

The simplicial structure is given by the function complexes $\mathbf{hom}(A,B)$, where $\mathbf{hom}(A,B)_n$ is the abelian group of homomorphisms

$$A \otimes R(\Delta^n) \to B.$$

If $A \to B$ is a cofibration of simplicial presheaves and $j : K \to L$ is a cofibration of simplicial sets, then the cofibration

$$(B \times K) \cup (A \times L) \subset B \times L$$

induces a cofibration

$$(R(B) \otimes R(K)) \cup (R(A) \otimes R(L)) \subset R(B) \otimes R(L)$$

which is a local weak equivalence if either $A \to B$ is a local weak equivalence or $K \to L$ is a weak equivalence of simplicial sets, by Lemma 8.1 and Corollary 8.4. It follows by a saturation argument that if $C \to D$ is a cofibration of $s\mathbf{Pre}_R$, then the map

$$(D \otimes R(K)) \cup (C \otimes R(L)) \to D \otimes R(L)$$

is a cofibration, which is a local weak equivalence if either $C \to D$ is a local weak equivalence or $K \to L$ is a weak equivalence of simplicial sets.

Left properness is proved with a comparison of long exact sequences in homology sheaves, which starts with the observation that every cofibration is a monomorphism. Right properness is automatic from the definition of injective fibration, and the corresponding property for simplicial presheaves.

The proof of statement 2), for simplicial sheaves of R-modules is completely analogous, and the verification of 3) follows the usual pattern.

We shall often write

$$A \otimes K = A \otimes R(K)$$

(sectionwise and degreewise tensor product) for a simplicial R-module A and a simplicial presheaf K.

The Dold-Kan correspondence

$$N : s\mathbf{Pre}_R \simeq \mathrm{Ch}_+(\mathbf{Pre}_R) : \Gamma.$$

induces an injective model structure on the category $\mathrm{Ch}_+(\mathbf{Pre}_R)$ of presheaves of chain complexes from the corresponding model structure on the category $s\mathbf{Pre}_R$ of simplicial modules given by Theorem 8.5.

A morphism $f : C \to D$ of $\mathrm{Ch}_+(\mathbf{Pre}_R)$ is said to be a *local weak equivalence* (respectively *cofibration*, *injective fibration*) if the induced map $f_* : \Gamma C \to \Gamma D$ is a local weak equivalence (respectively cofibration, injective fibration) of $s\mathbf{Pre}_R$.

Similar definitions are made for chain complexes in sheaves of R-modules, with respect to the model structure on sheaves of simplicial R-modules.

Then we have the following:

Corollary 8.6. *1) With these definitions, the category* $\mathrm{Ch}_+(\mathbf{Pre}_R)$ *of chain complexes in R-modules satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

2) With these definitions, the category $\mathrm{Ch}_+(\mathbf{Shv}_R)$ *of chain complexes in sheaves of R-modules satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

3) The inclusion and associated sheaf functors define a Quillen equivalence

$$L^2 : \mathrm{Ch}_+(\mathbf{Pre}_R) \leftrightarrows \mathrm{Ch}_+(\mathbf{Shv}_R) : i$$

between the (injective) model structures of parts 1) and 2).

Remark 8.7. Every injective fibration $p : C \to D$ of $\mathrm{Ch}_+(\mathbf{Pre}_R)$ corresponds to an injective fibration $p_* : \Gamma C \to \Gamma D$ of simplicial R-modules. The map p_* is a Kan fibration in each section (Lemma 5.12), so that the maps $p : C_n \to D_n$ are surjective in all sections for $n \geq 1$ [24, III.2.11]. The traditional identification of fibrations of simplical abelian groups with chain complex morphisms that are surjective in non-zero degrees fails for the injective model structures of Theorem 8.5 and Corollary 8.6. Chain complex morphisms $C \to D$ which are local epimorphisms in non-zero degrees correspond to local fibrations under the Dold-Kan correspondence.

The identification of cofibrant chain complexes with complexes of projective modules also fails for the injective model structures.

8.2 The derived category

Every ordinary chain complex C can be identified with an unbounded chain complex $C(0)$ by putting 0 in negative degrees. The right adjoint of the resulting functor is the *good truncation* $D \mapsto \mathrm{Tr}_0 D$ at level 0, where

$$\mathrm{Tr}_0 D_n = \begin{cases} \ker(\partial : D_0 \to D_{-1}) & \text{if } n = 0, \text{ and} \\ D_n & \text{if } n > 0. \end{cases}$$

If D is an unbounded complex and $n \in \mathbb{Z}$, then the shifted complex $D[n]$ is defined by

$$D[n]_p = D_{p+n}.$$

If C is an ordinary chain complex mand $n \in \mathbb{Z}$, define the shifted complex $C[n]$ by

$$C[n] = \mathrm{Tr}_0(C(0)[n]).$$

Suppose that $n > 0$. Then $C[-n]$ is the complex with $C[-n]_p = C_{p-n}$ for $p \geq n$ and $C[-n]_p = 0$ for $p < n$. Also, $C[n]$ is the complex with $C[n]_p = C_{p+n}$ for $p > 0$ and

$$C[n]_0 = \ker(\partial : C_n \to C_{n-1}).$$

There is, further, an adjunction isomorphism

$$\hom(C[-n], D) \cong \hom(C, D[n])$$

for all $n > 0$.

In particular, the functor $C \mapsto C[-1]$ is a suspension functor for ordinary chain complexes, while $C \mapsto C[1]$ is a loop functor. The suspension functor is left adjoint to the loop functor.

A *spectrum* D in chain complexes consists of chain complexes D^n, $n \geq 0$, together with chain complex maps

$$\sigma : D^n[-1] \to D^{n+1}$$

called *bonding morphisms* . A *morphism of spectra* $f : D \to E$ in chain complexes consists of chain complex maps $f : D^n \to E^n$ which respect structure in the sense that the diagrams

$$\begin{CD}
D^n[-1] @>{\sigma}>> D^{n+1} \\
@V{f[-1]}VV @VV{f}V \\
E^n[-1] @>>{\sigma}> E^{n+1}
\end{CD}$$

commute. We shall write $\mathbf{Spt}(\mathrm{Ch}_+(\ ))$ to denote the corresponding category of spectra , wherever it occurs. For example, $\mathbf{Spt}(\mathrm{Ch}_+(\mathbf{Pre}_R))$ is the category of spectra in chain complexes of R-modules.

Example 8.8. Suppose that E is an unbounded chain complex. There is a canonical map

$$\sigma : (\mathrm{Tr}_0\, E)[-1] \to \mathrm{Tr}_0(E[-1])$$

which is defined by the diagram

$$\begin{CD}
\vdots @. \vdots \\
@VVV @VVV \\
E_1 @>>> E_1 \\
@VVV @VVV \\
\ker(\partial) @>>> E_0 \\
@VVV @VVV \\
0 @>>> \ker(\partial)
\end{CD}$$

Replacing E by $E[-n]$ gives maps

$$\sigma : (\mathrm{Tr}_0(E[-n]))[-1] \xrightarrow{\mathrm{Tr}}_0 (E[-n-1]).$$

These are the bonding maps for a spectrum object $\mathrm{Tr}(E)$ with

$$\mathrm{Tr}(E)^n = \mathrm{Tr}_0(E[-n]).$$

Thus, every unbounded chain complex E defines a spectrum object $\mathrm{Tr}(E)$ in chain complexes.

Example 8.9. If C is a spectrum object in chain complexes, the maps

$$C^n(0)[-1] = C^n[-1](0) \to C^{n+1}(0)$$

have adjoints $C^n(0) \to C^{n+1}(0)[1]$ in the category of unbounded chain complexes. Write $C(0)$ for the colimit of the maps

$$C^0(0) \to C^1(0)[1] \to C^2(0)[2] \to \dots$$

in the unbounded chain complex category. Then it's not hard to see that $\mathrm{Tr}(C(0))^n$ is naturally isomorphic to the colimit of the diagram of chain complexes

$$C^n \to C^{n+1}[1] \to C^{n+2}[2] \to \dots$$

and that the adjoint bonding maps $\mathrm{Tr}(C(0))^n \to \mathrm{Tr}(C(0))^{n+1}[1]$ are the isomorphisms determined by the diagrams

$$\begin{tikzcd} C^n \arrow[r] \arrow[d] & C^{n+1}[1] \arrow[r] \arrow[d] & C^{n+2}[2] \arrow[r] \arrow[d] & \cdots \\ C^{n+1}[1] \arrow[r] & C^{n+2}[2] \arrow[r] & C^{n+3}[3] \arrow[r] & \cdots \end{tikzcd}$$

There is a canonical map

$$\eta : C \to \mathrm{Tr}(C(0)),$$

defined by maps to colimits. One usually writes

$$QC = \mathrm{Tr}(C(0)).$$

Lemma 8.10. *The suspension functor $C \mapsto C[-1]$ preserves cofibrations of chain complexes.*

Proof. It is enough to show that the functor $X \mapsto NR(X)[-1]$ takes cofibrations of simplicial presheaves X to cofibrations of $\mathrm{Ch}_+(\mathbf{Pre}_R)$.

But

$$R(X) = R_*(X_+),$$

where $R_*(X_+)$ is the reduced part of the complex $R(X_+)$ associated to $X_+ = X \sqcup \{*\}$, pointed by $*$. The functor $Y \mapsto R_*Y$ is left adjoint to the forgetful functor from $s\mathbf{Pre}_R$ to pointed simplicial presheaves, and therefore preserves cofibrations.

Also,

$$\overline{W}(R_*Y) \cong R_*(\Sigma Y),$$

where ΣY is the Kan suspension of the pointed simplicial presheaf Y, and the Kan suspension preserves cofibrations of pointed simplicial sets (or presheaves) [24, III.5]. The isomorphism

$$N(\overline{W}(R_*Y)) \cong NR_*Y[-1]$$

defines the simplicial R-module $\overline{W}(R_*Y)$.

Say that a map $f : E \to F$ of spectra in chain complexes is a *strict weak equivalence* (respectively *strict fibration*) if all maps $f : E^n \to F^n$ are local weak equivalences (respectively injective fibrations).

A *cofibration* is a map $i : A \to B$ of spectrum objects such that

1) the map $A^0 \to B^0$ is a cofibration of chain complexes, and
2) all induced maps

$$B^n[-1] \cup_{A^n[-1]} A^{n+1} \to B^{n+1}$$

are cofibrations.

It follows from Lemma 8.10 that if $i : A \to B$ is a cofibration of spectrum objects then all component maps $i : A^n \to B^n$ are cofibrations of chain complexes.

Lemma 8.11. *With the definitions of strict equivalence, strict fibration and cofibration given above, the category* $\mathbf{Spt}(\mathrm{Ch}_+(\mathbf{Pre}_R))$ *satisfies the axioms for a proper closed simplicial model category.*

The proof of Lemma 8.11 is a formality, and is a standard exercise from stable homotopy theory. The model structure of Lemma 8.11 is the *strict model structure* for the category of spectra in chain complexes.

Say that a map $f : A \to B$ of spectrum objects in chain complexes is a *stable equivalence* if the induced map $f_* : QA \to QB$ is a strict equivalence.

In view of the examples above, this means precisely that the induced map $f_* : A(0) \to B(0)$ of unbounded complexes is a stable equivalence if and only if it is a homology sheaf isomorphism. Also a map $g : E \to F$ of unbounded complexes induces a stable equivalence $g_* : \mathrm{Tr}(E) \to \mathrm{Tr}(F)$ if and only if g is a homology sheaf isomorphism.

A map $p : C \to D$ of spectrum objects is a *stable fibration* if and only if it has the right lifting property with respect to all maps which are cofibrations and stable equivalences.

Proposition 8.12. *The classes of cofibrations, stable equivalences and stable fibrations give the category* $\mathbf{Spt}(\mathrm{Ch}_+(\mathbf{Pre}_R))$ *the structure of a proper closed simplicial model category.*

Proof. The proof follows the "Bousfield-Friedlander script" [7], [24, X.4]. It is a formal consequence of the following assertions:

A1 The functor Q preserves strict weak equivalences.
A2 The maps η_{QC} and $Q(\eta_C)$ are strict weak equivalences for all spectrum objects C.
A3 The class of stable equivalences is closed under pullback along all stable fibrations, and is closed under pushout along all cofibrations.

Only the last of these statements requires proof, but it is a consequence of long exact sequence arguments in homology in the unbounded chain complex category. One uses Lemma 8.10 to show the cofibration statement. The fibration statement is proved by showing that every stable fibration $p : C \to D$ is a strict fibration, and so the induced map $C(0) \to D(0)$ of unbounded complexes is a local epimorphism in all degrees.

The model structure of Proposition 8.12 is the *stable model structure* for the category of spectrum objects in chain complexes of R-modules. The associated homotopy category

$$\mathrm{Ho}(\mathbf{Spt}(\mathrm{Ch}_+(\mathbf{Pre}_R)))$$

is the *derived category* for the category of presheaves (or sheaves) of R-modules.

8.3 Abelian sheaf cohomology

Suppose that C is a chain complex, with associated simplicial abelian object $\Gamma(C)$, and that X is a simplicial presheaf. Recall that the cocycle category $h(X,\Gamma(C))$ has for objects all pairs of maps

$$X \xleftarrow[\simeq]{f} U \xrightarrow{g} \Gamma(C).$$

The morphisms $(f,g) \to (f',g')$ of $h(X,\Gamma(C))$ are the commutative diagrams of simplicial set maps

$$\begin{tikzcd}
& U \arrow[dl, "f"'] \arrow[dd] \arrow[dr, "g"] & \\
X & & \Gamma(C) \\
& V \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

The category $h(X,\Gamma(C))$ is isomorphic, via adjunctions, to two other categories:

1) the category whose objects are all pairs

$$X \xleftarrow{f} U,\ \mathbb{Z}(U) \xrightarrow{g} \Gamma(C),$$

where $\mathbb{Z}(U)$ is the free simplicial abelian presheaf on U and g is a morphism of simplicial abelian presheaves, and

2) the category whose objects are all pairs

$$X \xleftarrow{f} U,\ N\mathbb{Z}(U) \xrightarrow{g} C,$$

where N is the normalized chains functor and g is a morphism of chain complexes.

Write $\pi(C,D)$ for the abelian group of chain homotopy classes of maps $C \to D$ between chain complexes C and D, and write $[\alpha]$ for the chain homotopy class of a morphism $\alpha : C \to D$.

There is a category $h_M(X,C)$ whose objects are all pairs

$$X \xleftarrow{f} U,\ \mathbb{Z}(U) \xrightarrow{[g]} C,$$

where $\mathbb{Z}(U)$ denotes the Moore complex associated to the simplicial abelian object $\mathbb{Z}(U)$ having the same name, and $[g]$ is a chain homotopy class of morphisms of chain complexes. A morphism $\theta : (f,[g]) \to (f',[g'])$ is a simplicial presheaf map θ which makes the diagrams

$$\begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dd, "\theta"] \\ X & \\ & V \arrow[ul, "f'"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}(U) \arrow[dr, "{[g]}"] \arrow[dd, "{[\theta_*]}"'] & \\ & C \\ \mathbb{Z}(V) \arrow[ur, "{[g']}"'] & \end{tikzcd}$$

commute.

Recall that there are natural chain maps $i : N\mathbb{Z}(U) \to \mathbb{Z}(U)$ and $p : \mathbb{Z}(U) \to N\mathbb{Z}(U)$ such that $p \cdot i$ is the identity on $N\mathbb{Z}(U)$ and that $i \cdot p$ is naturally chain homotopic to the identity on the Moore complex $\mathbb{Z}(U)$ [24, III.2.4].

The category $h_M(X,C)$ can then be identified up to isomorphism, via precomposition with the natural map i, with the category whose objects are the pairs

$$X \xleftarrow{f} U,\ N\mathbb{Z}(U) \xrightarrow{[g]} C,$$

and whose morphisms $\theta : (f,[g]) \to (f',[g'])$ are maps θ of simplicial presheaves such that the diagrams

$$\begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dd, "\theta"] \\ X & \\ & V \arrow[ul, "f'"] \end{tikzcd} \qquad \begin{tikzcd} N\mathbb{Z}(U) \arrow[dr, "{[g]}"] \arrow[dd, "{[\theta_*]}"'] & \\ & C \\ N\mathbb{Z}(V) \arrow[ur, "{[g']}"'] & \end{tikzcd}$$

commute.

Lemma 8.13. *Suppose given chain maps $\alpha,\beta : N\mathbb{Z}(U) \to C$ which are chain homotopic, and suppose that $f : U \to X$ is a local weak equivalence of simplicial presheaves. Then the cocycles (f,α) and (f,β) represent the same element of $\pi_0 h(X,\Gamma(C))$.*

Proof. Chain homotopies are defined by path objects for the projective model structure on the category of $\mathscr{C}^{op}$-diagrams (with sectionwise weak equivalences). Choose a projective cofibrant model $\pi : W \to U$ for this model structure. If there is a chain homotopy $\alpha \simeq \beta : N\mathbb{Z}(U) \to C$, then the composite maps $\alpha_*\eta\pi$ and $\beta_*\eta\pi$ are left homotopic for some choice of cylinder $W \otimes I$ for W in the projective model structure.

This means that there is a diagram

$$\begin{tikzcd} & W \arrow[r, "\pi"] \arrow[dl, "1"'] \arrow[d, "i_0"] & U \arrow[dr, "\alpha_*\eta"] & \\ W & W \otimes I \arrow[l, "s"'] \arrow[rr, "h"] & & \Gamma(C) \\ & W \arrow[r, "\pi"'] \arrow[ul, "1"] \arrow[u, "i_1"'] & U \arrow[ur, "\beta_*\eta"'] & \end{tikzcd}$$

where the maps s, i_0, i_1 are all part of the cylinder object structure for $W \otimes I$, and are sectionwise weak equivalences. It follows that

$$(f, \alpha_* \eta) \sim (f\pi, \alpha_* \eta\pi) \sim (f\pi s, h) \sim (f\pi, \beta_* \eta\pi) \sim (f, \beta_* \eta)$$

in $\pi_0 h(X, \Gamma(C))$.

As noted previously, we can identify $h(X, \Gamma(C))$ with the category of cocycles

$$X \xleftarrow{f} U,\ N\mathbb{Z}(U) \xrightarrow{\alpha} C,$$

where f is a local weak equivalence of simplicial presheaves and α is a chain map. Every such cocycle determines an object

$$X \xleftarrow{f} U,\ N\mathbb{Z}(U) \xrightarrow{[\alpha]} C,$$

of $h_M(X, C)$. This assignment is functorial, and therefore defines a functor

$$\psi : h(X, \Gamma(C)) \to h_M(X, C).$$

Lemma 8.14. *The functor $\psi : h(X, \Gamma(C)) \to h_M(X, C)$ induces an isomorphism*

$$[X, \Gamma(C)] \cong \pi_0 h(X, \Gamma(C)) \xrightarrow[\cong]{\psi_*} \pi_0 h_M(X, C).$$

Proof. If the chain maps $\alpha, \beta : N\mathbb{Z}(U) \to C$ are chain homotopic and $f : U \to X$ is a local weak equivalence, then the cocycles (f, α) and (f, β) are in the same path component of $h(X, \Gamma(C))$, by Lemma 8.13. The assignment

$$(f, [\alpha]) \mapsto [(f, \alpha)]$$

therefore defines a function

$$\gamma : \pi_0 h_M(X, C) \to \pi_0 h(X, \Gamma(C)),$$

and one checks that γ is the inverse of ψ_*.

Remark 8.15. If E is a simplicial abelian presheaf, then $E \cong \Gamma(NE)$, and the corresponding instance of the functor ψ has the form

$$h(X, E) \cong h(X, \Gamma(NE)) \xrightarrow{\psi} h_M(X, NE),$$

where the cocycle

$$X \xleftarrow[\simeq]{f} U \xrightarrow{\alpha} E$$

is sent to the object

$$X \xleftarrow{f} Y,\ \mathbb{Z}(U) \xrightarrow{p} N\mathbb{Z}(U) \xrightarrow{N\alpha_*} NE.$$

Here, $\alpha_* : \mathbb{Z}(U) \to E$ is the adjoint of the simplicial presheaf map $\alpha : U \to E$. The natural chain homotopy equivalence $i : NE \to E$ in the Moore chains of E induces an isomorphism

$$i_* : h_M(X,NE) \xrightarrow{\cong} h_M(X,E),$$

and the composite

$$h(X,E) \xrightarrow{\psi} h_M(X,NE) \underset{\cong}{\xrightarrow{i_*}} h_M(X,E) \tag{8.1}$$

takes the cocycle (f,α) to the object $(f,[\alpha_*])$, where $[\alpha_*]$ is the chain homotopy class of the induced map $\alpha_* : \mathbb{Z}(U) \to E$ of Moore complexes. The composite $i_*\psi$ is yet another variant of the functor ψ, and will be denoted by ψ.

The isomorphism

$$\pi_0 h_M(X,C) \cong [X,\Gamma(C)]$$

which results from Lemma 8.14 is a chain complex variant of the Verdier hypercovering theorem (Theorem 5.41). This result allows one to represent morphisms in the homotopy category taking values in simplicial abelian presheaves by chain homotopy classes of maps.

The simplicial presheaf $\Gamma(C)$ is locally fibrant. As in the proof of Theorem 5.41, there is a category $H_h(X,C)$ whose objects are pairs $([q],[\alpha])$ where $[q]$ is a simplicial homotopy class of a hypercover $q : U \to X$ and $[\alpha]$ is a chain homotopy class of a map $\alpha : \mathbb{Z}(U) \to C$. A morphism $([q],[\alpha]) \to ([q'],[\beta])$ in $H_h(X,C)$ is a simplicial homotopy class of maps $[\theta] : U \to V$ such that the diagrams

$$\begin{tikzcd} & U \arrow[dl, "{[q]}"'] \arrow[dd, "{[\theta]}"] \\ X & \\ & V \arrow[ul, "{[q']}"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}(U) \arrow[dr, "{[\alpha]}"] \arrow[dd, "{[\theta_*]}"'] & \\ & C \\ \mathbb{Z}(V) \arrow[ur, "{[\beta]}"'] & \end{tikzcd}$$

commute. There is a function

$$\omega : \pi_0 H_h(X,C) \to [X,\Gamma(C)]$$

which is defined by sending a class $[([q],[\alpha])]$ to the composite map $((\alpha \cdot i)_* \cdot \eta) \cdot q^{-1}$ in the homotopy category.

Recall that $h_{hyp}(X,\Gamma(C))$ is the full subcategory of $h(X,\Gamma(C))$ on those cocycles

$$X \xleftarrow{q} U \xrightarrow{\alpha} \Gamma(C)$$

such that the weak equivalence q is a hypercover. There is a functor

$$\gamma : h_{hyp}(X,\Gamma(C)) \to H_h(X,C)$$

which is defined by $(q,\alpha) \mapsto ([q],[\alpha_* \cdot p])$, and there is a commutative diagram

$$\begin{tikzcd} \pi_0 h_{hyp}(X,\Gamma(C)) \arrow[r, "\gamma_*"] \arrow[d, "\cong"'] & \pi_0 H_h(X,C) \arrow[d, "\omega"] \\ \pi_0 h(X,\Gamma(C)) \arrow[r, "\cong"'] & [X,\Gamma(C)] \end{tikzcd}$$

The function γ_* is plainly surjective, but it is also injective by the commutativity of the diagram. It follows that all functions in the diagram are bijections.

The resulting bijection

$$\pi_0 H_h(X,C) \cong [X,\Gamma(C)]$$

gives the following result:

Proposition 8.16. *Suppose that X is a simplicial presheaf and that C is a presheaf of chain complexes. Then there are isomorphisms*

$$[X,\Gamma(C)] \cong \pi_0 H_h(X,C) = \varinjlim_{[p]:U\to X} \pi(\mathbb{Z}(U),C).$$

Remark 8.17. The identification

$$[X,\Gamma(C)] \cong \varinjlim_{[p]:U\to X} \pi(\mathbb{Z}(U),C)$$

of Proposition 8.16 is an older form of Lemma 8.14, which appeared as Theorem 2.1 in [31]. The displayed colimit happens to be filtered by a standard calculus of fractions argument [9], but that observation is irrelevant for the proof which is given here.

Remark 8.18. It is useful in practice to have a more explicit method of representing the path component of a cocycle

$$X \xleftarrow[\simeq]{f} U,\ \mathbb{Z}/\ell(U) \xrightarrow{[\alpha]} C$$

in $h_M(X,C)$ by a cocyle $(p,[\beta])$ where $p: V \to X$ is a hypercover. In effect, form the composite simplicial presheaf map

$$U \xrightarrow{\eta} \mathbb{Z}/\ell(U) \xrightarrow{(\alpha\cdot i)_*} \Gamma(C)$$

where $(\alpha\cdot i)_*$ corresponds to the map

$$N\mathbb{Z}/\ell(U) \xrightarrow{i} \mathbb{Z}/\ell(U) \xrightarrow{\alpha} C$$

under the Dold-Kan correspondence. Then the simplicial abelian object $\Gamma(C)$ is locally fibrant, so that there is a diagram

$$\begin{tikzcd} & U \arrow[dl, "f"', "\simeq"] \arrow[dd, "j"] \arrow[dr, "(\alpha\cdot i)_*\eta"] & \\ X & & \Gamma(C) \\ & V \arrow[ul, "p", "\simeq"'] \arrow[ur, "g"'] & \end{tikzcd}$$

where p is a hypercover, as in Lemma 5.43. The morphisms i in the induced diagram

$$\begin{tikzcd} N\mathbb{Z}/\ell(U) \arrow[r, "i"] \arrow[dd, "j_*"'] & \mathbb{Z}/\ell(U) \arrow[dr, "\alpha"] & \\ & & C \\ N\mathbb{Z}/\ell(V) \arrow[r, "i"'] \arrow[urr, "g_*"] & \mathbb{Z}/\ell(V) \arrow[ur, dotted, "\beta"'] & \end{tikzcd}$$

are chain homotopy equivalences, so the dotted arrow β exists which makes the diagram commute up to chain homotopy. The chain homotopy class of β is also uniquely determined.

It follows that there are commutative diagrams

$$\begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dd, "j"] \\ X & \\ & V \arrow[ul, "p"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}/\ell(U) \arrow[dr, "[\alpha]"] \arrow[dd, "[j]"'] & \\ & C \\ \mathbb{Z}/\ell(V) \arrow[ur, "[\beta]"'] & \end{tikzcd}$$

so that the cocycle $(p,[\beta])$ is in the path component of $(f,[\alpha])$.

The point of this is that any finite list of cocycles $(f_1,[\alpha_1]),\dots,(f_n,[\alpha_n])$, where $\alpha_i : \mathbb{Z}/\ell(U_i) \to C_i$, have path component representatives $(p,[\beta_1]),\dots,(p,\beta_n)$ for $\beta_i : \mathbb{Z}/\ell(V) \to C_i$, where $p : V \to X$ is a *single choice of hypercover*. This is a consequence of the argument just displayed, together with the fact that hypercovers are preserved by pullback.

Recall that the abelian sheaf category on a small site has enough injectives (as does the abelian presheaf category): an abelian sheaf I is injective if and only if it has the right lifting property with respect to all inclusions of subobjects $B \subset \mathbb{Z}(U)$, $U \in \mathscr{C}$, so that one can show that there is is an inclusion $A \subset I$ with I injective by using a small object argument.

We shall identify cochain complexes with unbounded chain complexes which are concentrated in degrees $n \leq 0$.

Suppose that A is a sheaf of abelian groups, and let $A \to J$ be an injective resolution of A, where of course J is a cochain complex. Write $A[-n]$ for the chain complex consisting of A concentrated in degree n, and consider the chain map $A[-n] \to J[-n]$.

The simplicial abelian sheaf

$$K(A,n) = \Gamma A[-n]$$

defines the *Eilenberg-Mac Lane object* associated to A and n.

It is an abuse, but write

$$K(D,n) = \Gamma\,\mathrm{Tr}_0(D[-n])$$

for all chain complexes D, where $\mathrm{Tr}_0(D[-n])$ is the good truncation of $D[-n]$ in non-negative degrees. The simplicial abelian object $K(D,n)$ is not an Eilenberg-Mac Lane object in general.

There are isomorphisms

$$\pi(C(0), D[-n]) \cong \pi(C, \mathrm{Tr}_0\, D[-n]), \tag{8.2}$$

which are natural in ordinary chain complexes C and unbounded complexes D, where $C(0)$ is the unbounded complex which is constructed from C by putting 0 in all negative degrees.

Suppose that C is an ordinary chain complex and that K is a cochain complex. Form the bicomplex

$$\hom(C,K)_{p,q} = \hom(C_{-p}, K_q)$$

with the obvious induced differentials:

$$\begin{aligned}
\partial' &= \partial_C^* : \hom(C_{-p}, K_q) \to \hom(C_{-p-1}, K_q)\\
\partial'' &= (-1)^p \partial_{K*} : \hom(C_{-p}, K_q) \to \hom(C_{-p}, K_{q-1}).
\end{aligned}$$

Then $\hom(C,K)$ is a third quadrant bicomplex with total complex $\mathrm{Tot}_\bullet \hom(C,K)$ defined by

$$\begin{aligned}
\mathrm{Tot}_{-n} \hom(C,K) &= \bigoplus_{p+q=-n} \hom(C_{-p}, K_q)\\
&= \bigoplus_{0\leq p\leq n} \hom(C_p, K_{-n+p}),
\end{aligned}$$

for $n \geq 0$. The complex $\mathrm{Tot}_\bullet \hom(C,K)$ is concentrated in negative degrees.

Lemma 8.19. *There are natural isomorphisms*

$$H_{-n}(\mathrm{Tot}_\bullet \hom(C,K)) \cong \pi(C(0), K[-n]).$$

Proof. Write $(f_0, f_1, \dots, f_n)$ for a typical element of

$$\mathrm{Tot}_{-n} \hom(C,K) = \bigoplus_{0\leq p\leq n} \hom(C_p, K_{-n+p}).$$

Then

$$\partial(f_0, \dots, f_n) = (g_0, \dots, g_{n+1}),$$

where

$$g_k = \begin{cases} \partial f_0 & \text{if } k = 0, \\ f_{k-1}\partial + (-1)^k \partial f_k & \text{if } 0 < k < n+1, \text{ and} \\ f_n \partial & \text{if } k = n+1. \end{cases}$$

Set

$$\alpha(k) = \begin{cases} 1 & \text{if } k = 0, \text{ and} \\ \sum_{j=1}^{k-1} j & \text{if } k \geq 1. \end{cases}$$

Then the maps $(-1)^{\alpha(k)} f_k$ define a chain map $C \to K[-n]$.

Suppose that

$$\partial(s_0, \dots s_{n-1}) = (f_0, \dots, f_n).$$

Then the maps $(-1)^{\alpha(k)} s_k$ define a chain homotopy from the chain map $(-1)^{\alpha(k)} f_k$ to the 0 map.

Lemma 8.20. *Suppose that J is a cochain complex of injective sheaves, and that $f : C \to D$ is a homology isomorphism of ordinary chain complexes of presheaves. Then the induced morphism of cochain complexes*

$$\mathrm{Tot}_\bullet \hom(D, J) \xrightarrow{f^*} \mathrm{Tot}_\bullet \hom(C, J)$$

is a homology isomorphism.

Proof. The functors $\hom(\ , J_{-q})$ are exact, and there are isomorphisms

$$H_{-p} \hom(C, J_{-q}) \cong \hom(\tilde{H}_p(C), J_{-q}),$$

which are natural in chain complexes C. It follow that there is a spectral sequence with

$$E_1^{p,q} \cong \hom(\tilde{H}_p(C), J_{-q}) \Rightarrow \pi(C, \mathrm{Tr}_0 J[-p-q]) = H_{-p-q} \mathrm{Tot}_\bullet \hom(C, J).$$

which is natural in C. The claim follows from a comparison of such spectral sequences.

Corollary 8.21. *Suppose that J is a cochain complex of injective sheaves. Then every local weak equivalence $f : X \to Y$ of simplicial presheaves induces an isomorphism*

$$\pi(N\mathbb{Z}Y, \mathrm{Tr}_0 J[-n]) \xrightarrow{\cong} \pi(N\mathbb{Z}X, \mathrm{Tr}_0 J[-n])$$

in chain homotopy classes for all $n \geq 0$.

Again, let J be a cochain complex of injective sheaves. As in the proof of Lemma 8.14, there is a well defined abelian group homomorphism

$$\gamma : \pi(N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]) \to \pi_0 h(X, K(J, n))$$

which takes a chain homotopy class $[\alpha]$ to the element $[(1, \alpha_* \eta)]$, where $\alpha_* : \mathbb{Z}(X) \to K(J, n)$ is induced by α under the Dold-Kan correspondence, and $\eta : X \to \mathbb{Z}(X)$ is the adjunction map. This morphism is natural in simplicial presheaves X.

Lemma 8.22. *Suppose that J is a cochain complex of injective sheaves. Then we have the following:*

1) The map

$$\gamma : \pi(N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]) \to \pi_0 h(X, K(J,n)).$$

is an isomorphism.

2) The canonical map

$$\pi : \pi(N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]) \to [N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]]$$

is an isomorphism.

3) The simplicial abelian sheaf $K(J,n) = \Gamma(\mathrm{Tr}_0 J[-n])$ satisfies descent.

Recall that a simplicial presheaf X on a site $\mathscr{C}$ satisfies descent if some (hence any) injective fibrant model $j : X \to Z$ is a sectionwise weak equivalence in the sense that the simplicial set maps $j : X(U) \to Z(U)$ are weak equivalences for all objects U of $\mathscr{C}$.

Proof. For statement 1), suppose that $X \xleftarrow{f} Z \xrightarrow{g} K(J,n)$ is an object of $h(X, K(J,n))$. Then there is a unique chain homotopy class $[v] : N\mathbb{Z}X \to J[-n]$ such that $[v_* f] = [g]$ since f is a local weak equivalence, by Corollary 8.21. This chain homotopy class $[v]$ is also independent of the choice of representative for the path component of (f,g) in the cocycle category. We therefore have a well defined function

$$\omega : \pi_0 h(X, K(J,n)) \to \pi(N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]).$$

The composites $\omega \cdot \gamma$ and $\gamma \cdot \omega$ are identity morphisms.

For statement 2), observe that there is a commutative diagram

$$\begin{tikzcd}
\pi(N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]) \arrow[r, "\gamma", "\cong"'] \arrow[d, "\pi"'] & \pi_0 h(X, K(J,n)) \arrow[d, "\phi", "\cong"'] \\
{[N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]]} \arrow[r, "\cong"'] & {[X, K(J,n)]}
\end{tikzcd}$$

where ϕ is the isomorphism of Theorem 5.34, and the bottom isomorphism is induced by the Dold-Kan correspondence and the Quillen adjunction between simplicial presheaves and simplicial abelian presheaves.

Suppose that $j : \mathrm{Tr}_0 J[-n] \to C$ is an injective fibrant model for $K(J,n)$ in sheaves of chain complexes. Statement 3) says that the induced maps $j : \mathrm{Tr}_0 J[-n](U) \to C(U)$ of chain complexes are homology isomorphisms, for all $U \in \mathscr{C}$.

To prove this, first observe that there is a commutative diagram

$$\begin{tikzcd}
\pi(N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]) \arrow[r, "\pi", "\cong"'] \arrow[d, "j_*"'] & {[N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]]} \arrow[d, "j_*", "\cong"'] \\
\pi(N\mathbb{Z}X, C) \arrow[r, "\cong", "\pi"'] & {[N\mathbb{Z}X, C]}
\end{tikzcd}$$

for each simplicial presheaf X, in which the top occurrence of the canonical map π is an isomorphism by statement 2), and the bottom occurrence is an isomorphism since the chain complex object $N\mathbb{Z}X$ is cofibrant and C is injective fibrant. It follows that the map

$$j_* : \pi(N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]) \to \pi(N\mathbb{Z}X, C)$$

of chain homotopy classes is an isomorphism for all simplicial presheaves X. There is a split short sequence

$$0 \to N\mathbb{Z}* \to N\mathbb{Z}(\Delta^m/\partial\Delta^m) \to \mathbb{Z}[-m] \to 0$$

of chain complexes for all $m \geq 0$, and it follows that the maps

$$j_* : \pi(L_U(\mathbb{Z}[-m]), \mathrm{Tr}_0 J[-n]) \to \pi(L_U(\mathbb{Z}[-m]), C)$$

are isomorphisms for all $m, n \geq 0$ and $U \in \mathscr{C}$. Here, L_U is the left adjoint of the U-sections functor $E \mapsto E(U)$. The chain complex maps

$$\mathrm{Tr}_0 J[-n](U) \to C(U)$$

are therefore homology isomorphisms for all $U \in \mathscr{C}$, and statement 3) is proved.

The following result is a corollary of Lemma 8.22:

Theorem 8.23. *Suppose that A is a sheaf of abelian groups on $\mathscr{C}$, and let $A \to J$ be an injective resolution of A in the category of abelian sheaves. Let X be a simplicial presheaf on $\mathscr{C}$. Then there is an isomorphism*

$$\pi(N\mathbb{Z}X, \mathrm{Tr}_0 J[-n]) \cong [X, K(A,n)].$$

This isomorphism is natural in X.

Suppose that A is an abelian presheaf on $\mathscr{C}$ and that X is a simplicial presheaf. Write

$$H^n(X,A) = [X, K(A,n)],$$

and say that this group is the n^{th} *cohomology group* of X with coeffients in A.

The associated sheaf map

$$\eta : K(A,n) \to L^2K(A,n) \cong K(\tilde{A},n)$$

is a local weak equivalence, so there is a canonical isomorphism

$$H^n(X,A) \cong H^n(X,\tilde{A}).$$

Write

$$\tilde{H}_n(X,\mathbb{Z}) = \tilde{H}_n(N\mathbb{Z}X) \cong \tilde{H}_n(\mathbb{Z}X)$$

and call this object the n^{th} *integral homology sheaf* of the simplicial presheaf X. It is also common to write

$$\tilde{H}_n(X) = \tilde{H}_n(X,\mathbb{Z})$$

for the integral homology sheaves of X.

If A is an abelian presheaf, write

$$\tilde{H}_n(X,A) = \tilde{H}_n(N\mathbb{Z}(X)\otimes A) \cong \tilde{H}_n(\mathbb{Z}(X)\otimes A)$$

for the n^{th} *homology sheaf of X with coefficients in A.*

The following result gives a large class of examples:

Lemma 8.24. *Suppose that S is a simplicial object in $\mathscr{C}$ and that A is an abelian sheaf on $\mathscr{C}$. Then there are isomorphisms*

$$H^n(S,A) \cong H^n(\mathscr{C}/S, A|_S).$$

These isomorphisms are natural in abelian sheaves A.

Proof. Recall that $\mathscr{C}/S$ is the site fibred over the simplicial object S. This result is a consequence of Proposition 5.28.

Suppose that $j : K(A,n) \to GK(A,n)$ is an injective fibrant model on $\mathscr{C}$, and choose an injective fibrant model $GK(A,n)|_S \to W$ on the site $\mathscr{C}/S$. The Proposition 5.28 says that there is a weak equivalence

$$\mathbf{hom}(S, GK(A,n)) \simeq \mathbf{hom}(*,W).$$

The simplicial presheaf W is an injective fibrant model for the restricted simplicial presheaf $K(A|_S,n)$, and so there are isomorphisms

$$H^n(S,A) \cong \pi_0 \hom(S, GK(A,n)) \cong \pi_0 \mathbf{hom}(*,W) \cong H^n(\mathscr{C}/S, A|_S).$$

Remark 8.25. Both the statement of Lemma 8.24 and its proof are prototypical.

A similar argument shows that the étale cohomology group $H^n_{et}(S,A)$ of a simplicial T-scheme S, which is traditionally defined to be $H^n(et|_S, A|_S)$ for an abelian sheaf A on the big étale site [16], can be defined by

$$H^n_{et}(S,A) = [S, K(A,n)]$$

as morphisms in the injective homotopy category of simplical presheaves or sheaves on $(Sch|_T)_{et}$.

Here, $et|_S$ is the fibred étale site whose objects are the étale morphisms $\phi : U \to S_n$, and whose morphisms are diagrams of scheme homomorphisms of the form of (5.6), where the vertical maps are étale — this is usually what is meant by the étale site of a simplicial scheme S.

One uses the ideas of Example 5.27 to show that the restriction $Z|_S$ of an injective fibrant object Z to the site $et|_S$ satisfies descent. The remaining part of the argument for the weak equivalence

$$\mathbf{hom}(S,Z) \simeq \mathbf{hom}(*,W),$$

where $Z|_S \to W$ is an injective fibrant model on $et|_S$, is formal.

A different argument is available for the étale cohomological analogue of Corollary 8.24 if one's sole interest is a cohomology isomorphism: see [31].

Analogous techniques and results are available for other standard algebraic geometric topologies, such as the flat or Nisnevich topologies.

The following result is a consequence of Lemma 8.19 and Theorem 8.23:

Corollary 8.26. *Suppose that X is a simplicial presheaf and that A is a presheaf of abelian groups. Then there is a spectral sequence, with*

$$E_2^{p,q} = \operatorname{Ext}^q(\tilde{H}_p(X,\mathbb{Z}),\tilde{A}) \Rightarrow H^{p+q}(X,A). \tag{8.3}$$

The spectral sequence (8.3) is the *universal coefficients spectral sequence* for the cohomology groups $H^*(X,A)$.

Proof. Suppose that $A \to J$ is an injective resolution of A and let C be an ordinary chain complex of presheaves. The bicomplex $\hom(C,J)$ determines a spectral sequence with

$$E_2^{p,q} = \operatorname{Ext}^q(\tilde{H}_pC,A) \Rightarrow \pi(C,\operatorname{Tr}_0 J[-p-q]), \tag{8.4}$$

by Lemma 8.19. The special case $C = N\mathbb{Z}(X)$ is the required spectral sequence (8.3), by Theorem 8.23.

Example 8.27. Suppose that X is a simplicial set and that A is an abelian sheaf on a small site $\mathscr{C}$. The cohomology $H^*(\Gamma^*X,A)$ of the constant simplicial presheaf Γ^*X with coefficients in A is what Grothendieck would call a *mixed cohomology theory* [25]. In this case, the universal coefficients spectral sequence has a particularly simple form, in that there is a short exact sequence

$$\begin{aligned} 0 \to \bigoplus_{p+q=n} \operatorname{Ext}^1(H_{p-1}(X,\mathbb{Z}),H^q(\mathscr{C},A)) &\to H^{p+q}(\Gamma^*X,A) \\ &\to \bigoplus_{p+q=n} \hom(H_p(X,\mathbb{Z}),H^q(\mathscr{C},A)) \to 0. \end{aligned}$$

The existence of this sequence is best proved with the standard argument that leads to the classical universal coefficients theorem: apply the functor $\hom(\ ,\Gamma_*I)$ to the short split exact sequence of chain complexes

$$0 \to Z(\mathbb{Z}X) \to \mathbb{Z}X \to B(\mathbb{Z}X)[-1] \to 0.$$

where the complexes $Z(\mathbb{Z}X)$ and $B(\mathbb{Z}X)$ consist of cycles and boundaries, respectively, with 0 differentials.

Suppose that R is a presheaf of commutative rings with unit. There are R-modules versions of all results so far encountered in this section. In particular, there is an R-linear universal coefficients spectral sequence:

Lemma 8.28. *Suppose that X is a simplicial presheaf and that A is a presheaf of R-modules. Then there is a spectral sequence, with*

$$E_2^{p,q} = \mathrm{Ext}_R^q(\tilde{H}_p(X,\tilde{R}),\tilde{A}) \Rightarrow H^{p+q}(X,A). \tag{8.5}$$

We also have the following R-linear analog of Corollary 8.21:

Corollary 8.29. *Suppose that the simplicial presheaf map $f : X \to Y$ induces a homology sheaf isomorphism*

$$\tilde{H}_*(X,R) \cong \tilde{H}_*(Y,R).$$

Then f induces an isomorphism

$$H^*(Y,A) \to H^*(X,A)$$

for all presheaves of R-modules A.

Corollary 8.29 can also be proved with a comparison of the universal coefficients spectral sequences of Lemma 8.28.

The *sheaf cohomology group* $H^n(\mathscr{C},A)$ for an abelian sheaf A on a site $\mathscr{C}$ is traditionally defined by

$$H^n(\mathscr{C},A) = H_{-n}(\Gamma_* J)$$

where $A \to J$ is an injective resolution of A concentrated in negative degrees and Γ_* is the global sections functor (ie. inverse limit). But $\Gamma_* Y = \hom(*,Y)$ for any Y, where $*$ is the one-point simplicial presheaf, and so there are isomorphisms

$$H^n(\mathscr{C},A) \cong \pi(\mathbb{Z}*, \mathrm{Tr}_0 J[-n]) \cong [*, K(A,n)]$$

by Theorem 8.23. We have proved

Theorem 8.30. *Suppose that A is an abelian sheaf on a site $\mathscr{C}$. Then there is an isomorphism*

$$H^n(\mathscr{C},A) \cong [*, K(A,n)]$$

which is natural in abelian sheaves A.

We end this section with a seminal calculation:

Proposition 8.31. *Suppose that A is a presheaf of abelian groups, and that X is a simplicial presheaf. Suppose that the map*

$$j : K(A,n) \to GK(A,n)$$

is an injective fibrant model of $K(A,n)$. Then there are isomorphisms

$$\pi_j \mathbf{Hom}(X, GK(A,n))(U) \cong \begin{cases} H^{n-j}(X|_U, A|_U) & 0 \le j \le n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathscr{C}$.

Proof. Recall that $\mathbf{Hom}(X,Y)$ is the internal function complex (5.5).

There are isomorphisms

$$\pi_0\mathbf{Hom}(X,GK(A,n))(U) \cong [X|_U, GK(A|_U,n)] \cong H^n(X|_U, A|_U),$$

since $GK(A,n)|_U$ is an injective fibrant model of $K(A|_U,n)$ by Corollary 5.25 and Theorem 8.30.

The associated sheaf map

$$\eta : K(A,0) \to K(\tilde{A},0)$$

is an injective fibrant model for the constant simplicial presheaf $K(A,0)$ by Lemma 5.11, and there is an isomorphism

$$\mathbf{Hom}(X,K(\tilde{A},0)) \cong \mathbf{Hom}(\tilde{\pi}_0(X),\tilde{A}),$$

where the latter is identified with a constant simplicial sheaf. It follows that the sheaves $\tilde{\pi}_j\mathbf{Hom}(X,K(\tilde{A},0))$ vanish for $j > 0$.

There is a sectionwise fibre sequence

$$K(A,n-1) \to WK(A,n-1) \xrightarrow{p} K(A,n)$$

where the simplicial abelian presheaf $WK(A,n-1)$ is sectionwise contractible.

Take an injective fibrant model

$$\begin{tikzcd} WK(A,n-1) \arrow[r, "j"] \arrow[d, "p"'] & GWK(A,n-1) \arrow[d, "q"] \\ K(A,n) \arrow[r, "j"'] & GK(A,n) \end{tikzcd}$$

for the map p. This means that the maps labelled j are local weak equivalences, $GK(A,n)$ is injective fibrant and q is an injective fibration. Let $F = q^{-1}(0)$. Then F is injective fibrant and the induced map

$$K(A,n-1) \to F$$

is a local weak equivalence, by Lemma 4.30. Write $GK(A,n-1)$ for F.

We have injective (hence sectionwise) fibre sequences

$$\mathbf{Hom}(X,GK(A,n-1)) \to \mathbf{Hom}(X,GWK(A,n-1)) \to \mathbf{Hom}(X,GK(A,n))$$

by Lemma 5.12 and the enriched simplicial model structure of Corollary 5.19. The map

$$\mathbf{Hom}(X,GWK(A,n-1)) \to \mathbf{Hom}(X,*) \cong *$$

is a trivial injective fibration, and is therefore a sectionwise trivial fibration. It follows that there are isomorphisms

$$\pi_j \mathbf{Hom}(X, GK(A,n))(U) \cong \pi_{j-1} \mathbf{Hom}(X, GK(A,n-1))(U)$$

for $j \geq 1$ and all $U \in \mathscr{C}$, so that

$$\pi_j \mathbf{Hom}(X, GK(A,n))(U) \cong H^{n-j}(X|_U, \tilde{A}|_U)$$

for $1 \leq j \leq n$ and $\pi_j \mathbf{Hom}(X, GK(A,n))(U) = 0$ for $j > n$, by induction on n.

Corollary 8.32. *Suppose that A is a presheaf of abelian groups, and that*

$$j : K(A,n) \to GK(A,n)$$

is an injective fibrant model of $K(A,n)$. Then there are isomorphisms

$$\pi_j GK(A,n)(U) \cong \begin{cases} H^{n-j}(\mathscr{C}/U, \tilde{A}|_U) & 0 \leq j \leq n \\ 0 & j > n. \end{cases}$$

for all $U \in \mathscr{C}$.

8.4 Products and pairings

The category of *pointed simplicial presheaves* on a site $\mathscr{C}$ is the slice category $*/s\mathbf{Pre}(\mathscr{C})$. The objects can alternatively be viewed as pairs (X,x), where X is a simplicial presheaf and x is a choice of vertex in the global sections simplicial set

$$\Gamma_* X = \varprojlim_{U \in \mathscr{C}} X(U).$$

A *pointed map* $f : (X,x) \to (Y,y)$ is a simplicial presheaf map $f : X \to Y$ such that $f_*(x) = y$ in global sections, or equivalently such that the diagram

$$\begin{tikzcd} & * \arrow[dl, "x"'] \arrow[dr, "y"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes. I also write $s\mathbf{Pre}_*(\mathscr{C})$ to denote this category.

All slice categories for $s\mathbf{Pre}(\mathscr{C})$ inherit injective model structures from the injective model structure for simplicial presheaves — see Remark 5.45. In the case at hand, a pointed map $(X,x) \to (Y,y)$ is a local weak equivalence (respectively cofibration, injective fibration) if the underlying map $f : X \to Y$ is a local weak equivalence (respectively cofibration, injective fibration) of simplicial presheaves.

One writes $[X,Y]_*$ for morphisms in the pointed homotopy category

$$\mathrm{Ho}(s\mathbf{Pre}_*(\mathscr{C})).$$

The functor $q : s\mathbf{Pre}_*(\mathscr{C}) \to s\mathbf{Pre}(\mathscr{C})$ forgets the base point. One usually just writes $Y = q(Y)$ for the underlying simplicial presheaf of an object Y. The left adjoint of this functor $X \mapsto X_+$ is defined by adding a disjoint base point: $X_+ = X \sqcup \{*\}$. The functor q and its left adjoint form a Quillen adjunction, and there is a bijection

$$[X_+,Y]_* \cong [X,Y].$$

Every simplicial abelian presheaf B is canonically pointed by 0, so there is an isomorphism

$$[X_+,B]_* \cong [X,B].$$

In particular, cohomology groups can be computed in the pointed homotopy category via the natural isomorphism

$$H^n(X,A) = [X,K(A,n)] \cong [X_+,K(A,n)]_*.$$

The *smash product* $X \wedge Y$ of two pointed simplicial presheaves is formed just as in simplicial sets:

$$X \wedge Y = (X \times Y)/(X \vee Y)$$

where the wedge $X \vee Y$ is the coproduct of X and Y in the pointed category.

Suppose that A is a presheaf of abelian groups, and write $S^n \otimes A$ for the simplicial abelian presheaf $\mathbb{Z}(S^n) \otimes A$. Here, S^n is the n-fold smash power

$$S^n = S^1 \wedge \cdots \wedge S^1$$

of the simplicial circle $S^1 = \Delta^1/\partial\Delta^1$.

If X is a pointed simplicial presheaf, write $\tilde{\mathbb{Z}}(X)$ for the cokernel of the map

$$\mathbb{Z}(*) \to \mathbb{Z}(X)$$

which is defined by the base point of X. The homology sheaves

$$\tilde{H}_*(\tilde{\mathbb{Z}}(X) \otimes A)$$

are the *reduced homology sheaves* of X with coefficients in the abelian presheaf A.

The isomorphism

$$\mathbb{Z}(X) \otimes \mathbb{Z}(Y) \xrightarrow{\cong} \mathbb{Z}(X \times Y)$$

of simplicial abelian presheaves induces an isomorphism

$$T : \tilde{\mathbb{Z}}(X) \otimes \tilde{\mathbb{Z}}(Y) \xrightarrow{\cong} \tilde{\mathbb{Z}}(X \wedge Y) \tag{8.6}$$

which is natural in pointed simplicial presheaves X and Y.

The simplicial abelian presheaf $\tilde{\mathbb{Z}}(S^n)\otimes A$ has a unique homology presheaf, namely

$$H_n(\tilde{\mathbb{Z}}(S^n)\otimes A)\cong A,$$

and the good truncation functor Tr_n in chain complexes defines homology presheaf isomorphisms

$$\tilde{\mathbb{Z}}(S^n)\otimes A \xleftarrow{\simeq} \mathrm{Tr}_n(\tilde{\mathbb{Z}}(S^n)\otimes A) \xrightarrow{\simeq} \Gamma(A[-n]).$$

It follows that the simplicial abelian presheaf $\tilde{\mathbb{Z}}(S^n)\otimes A$ is naturally locally equivalent to the Eilenberg-Mac Lane object $K(A,n)$.

The natural isomorphism (8.6) induces a natural isomorphism

$$T_* : (\tilde{\mathbb{Z}}(S^n)\otimes A)\otimes(\tilde{\mathbb{Z}}(S^m)\otimes B) \xrightarrow{\cong} \tilde{\mathbb{Z}}(S^{n+m})\otimes(A\otimes B), \tag{8.7}$$

which gives a pairing

$$(\tilde{\mathbb{Z}}(S^n)\otimes A)\wedge(\tilde{\mathbb{Z}}(S^m)\otimes B) \to (\tilde{\mathbb{Z}}(S^n)\otimes A)\otimes(\tilde{\mathbb{Z}}(S^m)\otimes B) \xrightarrow[\cong]{T_*} \tilde{\mathbb{Z}}(S^{n+m})\otimes(A\otimes B)$$

of pointed simplicial presheaves. This pairing can be rewritten as a map

$$\cup : K(A,n)\wedge K(B,m) \to K(A\otimes B, n+m) \tag{8.8}$$

in the pointed homotopy category. This pairing, in any of its equivalent forms, is the *cup product* pairing. It induces the *external cup product*

$$\cup : H^n(X,A)\times H^m(Y,B) \to H^{n+m}(X\times Y, A\otimes B), \tag{8.9}$$

by definition of the cohomology groups.

The external cup product has an explicit description in terms of cocycles. Suppose that E and F are presheaves of simplicial abelian groups. Then there is a natural map

$$\cup : E\wedge F \to E\otimes F$$

of pointed simplicial presheaves which takes values in the degreewise tensor product. Given cocycles

$$X \xleftarrow[\simeq]{u} U \xrightarrow{f} E, \quad Y \xleftarrow[\simeq]{v} V \xrightarrow{g} F,$$

there is a cocycle

$$X\times Y \xleftarrow{u\times v} U\times V \xrightarrow{(f\wedge g)_*} E\otimes F,$$

where $(f\wedge g)_*$ is the composite

$$U\times V \to (U\times V)_+ \cong U\wedge V \xrightarrow{f\wedge g} E\wedge F \xrightarrow{\cup} E\otimes F.$$

The assignment

$$((f,u),(g,v)) \mapsto ((f\wedge g)_*, u\times v)$$

is functorial in the cocycles (f,u) and (g,v), and defines a functor

$$h(X,E) \times h(Y,F) \to h(X \times Y, E \otimes F).$$

The induced map in path components gives the cup product pairing

$$\cup : [X,E] \times [Y,F] \to [X \times Y, E \otimes F].$$

The cup product, as we've defined it, is derived from tensor products of simplicial abelian groups. There is an alternative approach which is based on the tensor product of chain complexes.

By Lemma 8.14, an element of $[X,E]$ can be represented by a cocycle

$$X \underset{\simeq}{\xleftarrow{u}} U,\ \mathbb{Z}(U) \xrightarrow{[\alpha]} E,$$

where $[\alpha]$ is the chain homotopy class of a map $\alpha : \mathbb{Z}(U) \to E$ of Moore complexes. If we have a second cocycle

$$Y \underset{\simeq}{\xleftarrow{v}} V,\ \mathbb{Z}(V) \xrightarrow{[\beta]} F$$

then the cocycle

$$X \times Y \underset{\simeq}{\xleftarrow{u \times v}} U \times V, \quad \mathbb{Z}(U \times V) \underset{\simeq}{\xrightarrow{f}} \mathbb{Z}(U) \otimes_{ch} \mathbb{Z}(V) \xrightarrow{[\alpha \otimes \beta]} E \otimes_{ch} F \tag{8.10}$$

represents an element $[(u \times v, [\alpha \otimes \beta])]$ of $\pi_0 h_M(X \times Y, E \otimes_{ch} F)$, where the natural equivalence of chain complexes f is the Alexander-Whitney map and the tensor products are in the chain complex category.

The chain homotopy class $[\alpha \otimes \beta]$ may not be independent of the chain homotopy classes of α and β, unless U and V are projective cofibrant objects. We can nevertheless always refine U and V by projective cofibrant models $p : U' \to U$ and $q : V' \to V$, and the cocycle

$$X \times Y \xleftarrow{u \cdot p \times v \cdot q} U' \times V',\ [\alpha \cdot p_* \otimes \beta \cdot q_*] : \mathbb{Z}(U') \otimes_{ch} \mathbb{Z}(V') \to E \otimes_{ch} F$$

is in the path component of the cocycle $(u \times v, [\alpha \otimes \beta])$ in $\pi_0 h_M(X \times Y, E \otimes_{ch} F)$.

It follows that the assignment

$$([(u,[\alpha])],[(v,[\beta])]) \mapsto [(u \times v, [\alpha \otimes \beta])]$$

defines a natural function

$$c : \pi_0 h_M(X,E) \times \pi_0 h_M(Y,F) \to \pi_0 h_M(X \times Y, E \otimes_{ch} F),$$

or equivalently a pairing

$$\cup_{ch} : [X,E] \times [Y,F] \to [X \times Y, \Gamma(E \otimes_{ch} F)].$$

The following result implies that the pairing $\cup_{ch}$ which is derived from the chain complex tensor product is naturally isomorphic to the cup product pairing:

Lemma 8.33. *The diagram*

$$\begin{tikzcd}
\pi_0 h_M(X,E)\times \pi_0 h_M(Y,F) \arrow[r, "c"] & \pi_0 h_M(X\times Y, E\otimes_{ch} F) \\
 & \pi_0 h_M(X\times Y, E\otimes F) \arrow[u, "\cong"', "f_*"] \\
{[X,E]\times[Y,F]} \arrow[r, "\cup"'] \arrow[uu, "\psi_*\times\psi_*", "\cong"'] & {[X\times Y, E\otimes F]} \arrow[u, "\cong"', "\psi_*"]
\end{tikzcd}$$

commutes, where ψ_* is the isomorphism of Remark 8.15, and $f : E\otimes F \to E\otimes_{ch} F$ is the Eilenberg-Zilber equivalence.

Proof. The functor ψ takes a cocycle (u,α) to the cocycle $(u,[\alpha_*])$, where the chain map $\alpha_* : \mathbb{Z}(U)\to E$ is the map of Moore complexes associated to the map α_* of simplicial abelian presheaves. The commutativity of the diagram is an easy consequence of the naturality of the Eilenberg-Zilber equivalence [24, IV.2.4].

If $X = Y$ and $A = B$ is a presheaf of commutative rings with unit, then precomposition with the diagonal $\Delta : X \to X\times X$ and composition with the multiplication $A\otimes A \to A$, applied to the pairing (8.9), together define the pairing

$$H^n(X,A)\times H^m(X,A)\to H^{n+m}(X\times X, A\otimes A)\to H^{n+m}(X,A), \tag{8.11}$$

which is the cup product for the cohomology of the simplicial presheaf X with coefficients in the presheaf of rings A.

The cup product ring structure on $H^*(X,A)$ is associative, and has two-sided multiplicative identity which is defined by the composite

$$X \to * \xrightarrow{1} A,$$

where the global section 1 is the multiplicative identity of the presheaf of rings A. The resulting ring structure on the cohomology $H^*(X,A)$ is graded commutative, since A is commutative and the twist isomorphism

$$S^p\wedge S^q \xrightarrow[\cong]{\tau} S^q\wedge S^p$$

is multiplication by $(1-)^{pq}$ in the homotopy category. In particular, the cohomology $H^*(X,A)$ of a simplicial presheaf X with coefficients in a commutative unitary ring A has the structure of a graded commutative ring.

Remark 8.34. Note the level of generality. Cup products are defined for cohomology of simplicial presheaves having all abelian presheaf coefficients, on all Grothendieck sites. It is an exercise to show that cup products are preserved by inverse image functors associated with geometric morphisms.

At one time, cup products were only defined, via Godement resolutions, in toposes having enough points — see [53].

8.5 Künneth formulas

The universal coefficients spectral sequence (8.3) admits substantial generalization.

In particular, suppose that D is a cochain complex of sheaves, and let $D \to I$ be a Cartan-Eilenberg resolution. This means, in part, that I is a bicomplex of injectives $I_{p,q}$ where $p,q \leq 0$ and the maps $D_p \to I_{p,q}$ form a bicomplex map such that the induced map $D \to \mathrm{Tot}_\bullet I$ of cochain complexes is a homology sheaf isomorphism.

The resolution I is constructed by inductively constructing resolutions of all of the exact sequences making up the cochain complex D, all of which are degreewise split. The construction has the following important features:

1) Write $B_{p,q} = B_p(I_{*,q})$, $Z_{p,q} = Z_p(I_{*,q})$ and $H_{p,q} = H_p(I_{*,q})$. Then the maps $B_p(D) \to B_{p,*}$, $Z_p(D) \to Z_{p,*}$ and $H_p(D) \to H_{p,*}$ all define injective resolutions.
2) The maps
$$I_{p,q} \twoheadrightarrow B_{p-1,q} \hookrightarrow Z_{p-1,q} \hookrightarrow I_{p-1,q} \tag{8.12}$$
making up the boundary morphism $I_{p,q} \to I_{p-1,q}$ are split, for each p and q.

Choose a Cartan-Eilenberg resolution $D_n \to I_{n,*}$ of D, and consider the tricomplex $\hom(X_n, I_{p,q})$. Computing homology in the p-direction gives the bicomplex $\hom(X_n, H_{p,*})$, since all maps (8.12) in the bicomplex $I_{*,*}$ are split, and it follows that there is a spectral sequence with

$$E_2^{p,q} = H^q(X, H_{-p}(D)) \Rightarrow [X, K(D, p+q)]. \tag{8.13}$$

The spectral sequence (8.13)is a generalized *hypercohomology* spectral sequence. It specializes to the standard hypercohomology spectral sequence if X is a point: the hypercohomology group $H^n(\mathscr{C}, D)$ is usually defined to be the cohomology group $H^n(\Gamma_* \mathrm{Tot}(I))$.

In all that follows, if X is a simplicial presheaf and D is a cochain complex of presheaves, then $\mathbf{Hom}(X,D)$ is the third quadrant presheaf of bicomplexes with

$$\mathbf{Hom}(X,D)_{-p,-q} = \mathbf{Hom}(X_p, D_{-q}),$$

where the coboundary in the p variable is induced by the boundary in the Moore complex for X. By the usual exponential law (see 5.6), there is an isomorphism of of bicomplexes
$$\mathbf{Hom}(X,D)(U) \cong \hom(X \times U, D) \tag{8.14}$$
for all $U \in \mathscr{C}$, which gives an alternative definition of $\mathbf{Hom}(X,D)$.

If I is a cochain complex of injective sheaves, then the sheaves $\mathbf{Hom}(Y_{-p}, I_q)$ in bidegree (p,q) are injectives, since there are natural isomorphisms

$$\hom(A, \mathbf{Hom}(Y_{-p}, I_q) \cong \hom(A \otimes \mathbb{Z}Y_{-p}, I_q),$$

and the functor $A \mapsto A \otimes \mathbb{Z}Y_{-p}$ preserves monomorphisms in A.

Suppose that $A \to J$ is an injective resolution of an abelian sheaf A, and write $\tilde{H}^q(Y,A)$ for the cohomology sheaves of the bicomplex $\mathbf{Hom}(Y,J)$. The sheaf $\tilde{H}^q(Y,A)$ is the sheaf associated to the presheaf which is defined by

$$U \mapsto H^q(Y \times U, A).$$

If X is a second choice of simplicial presheaf, then there is a tricomplex isomorphism

$$\hom(X, \mathbf{Hom}(Y,J)) \cong \hom(X \times Y, J),$$

and one has isomorphisms

$$H^n \operatorname{Tot}(\hom(X \times Y, J)) \cong H^n(X \times Y, A).$$

If $j : \mathbf{Hom}(Y,J) \to I$ is a Cartan-Eilenberg resolution, then j is a weak equivalence of cochain complexes of injectives in total complexes, and therefore induces isomorphisms

$$\pi(\mathbb{Z}X, \operatorname{Tr}_0 \operatorname{Tot}(\mathbf{Hom}(Y,J))[-n]) \xrightarrow{\cong} \pi(\mathbb{Z}X, \operatorname{Tr}_0 \operatorname{Tot}(I)[-n])$$

for all $n \geq 0$, by Lemma 8.22. It follows that the tricomplex map

$$\hom(X, \mathbf{Hom}(Y,J)) \to \hom(X, I)$$

induces a homology isomorphism of total complexes. We therefore obtain a spectral sequence

$$E_2^{p,q} = H^q(X, \tilde{H}^p(Y,A)) \Rightarrow H^{p+q}(X \times Y, A), \tag{8.15}$$

which is called the *Künneth spectral sequence*.

The special case

$$E_2^{p,q} = H^q(\mathscr{C}, \tilde{H}^p(Y,A)) \Rightarrow H^{p+q}(Y,A) \tag{8.16}$$

of the Künneth spectral sequence corresponding to letting X be the terminal object is a simple form of the *descent spectral sequence* for the cohomology of Y with coefficients in the abelian sheaf A. There are more delicate versions of the descent spectral sequence which exist in non-abelian settings, and **will be discussed later**.

Suppose that S is a scheme and that ℓ is a prime number. Say that a simplicial presheaf Z on $(Sch|_S)_{et}$ is *cohomologically smooth and proper* (with respect to ℓ-torsion sheaves) if the sheaves $\tilde{H}^q(Z, \mathbb{Z}/\ell)$ on the étale site $(Sch|_S)_{et}$ are locally constant and finite.

The assertion that a sheaf F on $(Sch|_S)_{et}$ is locally constant and finite means that there is an étale covering family $\phi : U \to S$ such that the sheaf $\phi^*(F)$ is constant and finite for all members ϕ of the covering. This is equivalent to the assertion that for all maps $p : \mathrm{Sp}(\mathscr{O}^{sh}_{x,S}) \to S$ arising from points $x \in S$ the sheaf $p^*(F)$ is constant and finite on $(Sch|_{\mathscr{O}^{sh}_{x,S}})_{et}$. Here, $\mathscr{O}^{sh}_{x,S}$ is the strict henselization of the local ring $\mathscr{O}_{x,S}$ at x.

In particular, if $S = \mathrm{Sp}(\mathscr{O})$ for a strict local hensel ring $\mathscr{O}$ then a sheaf F on the big étale site $(Sch|_{\mathscr{O}})_{et}$ is locally constant and finite if and only if it is constant and finite. This follows from the fact that every étale covering $U \to \mathrm{Sp}(\mathscr{O})$ has a section.

Observe that every separably closed field k is a strict local hensel ring.

Example 8.35. If ℓ is distinct from the residue characteristics of S and the scheme homomorphism $X \to S$ is smooth and proper, then the (degreewise constant) simplicial sheaf represented by X is cohomologically smooth and proper, by the smooth proper base change theorem [12, IV.1.1], [53, VI.4.2].

Example 8.36. Suppose that $G_S \to S$ is a group-scheme which is defined by a Chevalley integral form $G_{\mathbb{Z}}$, and suppose that ℓ is distinct from the residue characteristics of S.

Friedlander and Parshall show [18, Prop. 7] that the induced map

$$H^*(G_{\mathscr{O}}, \mathbb{Z}/\ell) \to H^*(G_k, \mathbb{Z}/\ell)$$

is an isomorphism for all geometric points $\mathrm{Sp}(k) \to \mathrm{Sp}(\mathscr{O})$, if $\mathscr{O}$ is a strict local hensel ring such that $1/\ell \in \mathscr{O}$. The statement includes all residue maps

$$H^*(G_{\mathscr{O}^{sh}_{x,X}}, \mathbb{Z}/\ell) \to H^*(G_{\overline{k(x)}}, \mathbb{Z}/\ell)$$

associated to points $x \in X \to \mathrm{Sp}(\mathscr{O})$, so that all specialization maps

$$H^*(G_{\mathscr{O}}, \mathbb{Z}/\ell) \to H^*(G_{\mathscr{O}^{sh}_{x,X}}, \mathbb{Z}/\ell)$$

are isomorphisms, and the sheaves $\tilde{H}^*(G_{\mathscr{O}}, \mathbb{Z}/\ell)$ on $(Sch|_{\mathscr{O}})_{et}$ are constant.

For any algebraically closed field k there is a geometric point $x : \mathrm{Sp}(k) \to \mathrm{Sp}(\mathbb{Z})$ and corresponding geometric points

$$\mathrm{Sp}(k) \to \mathrm{Sp}(\mathscr{O}^{sh}_{x,\mathbb{Z}}) \leftarrow \mathrm{Sp}(\mathbb{C}).$$

The Friedlander-Parshall theorem implies that the induced maps

$$H^*(G_k, \mathbb{Z}/\ell) \leftarrow H^*(G_{\mathscr{O}^{sh}_{x,\mathbb{Z}}}, \mathbb{Z}) \to H^*(G_{\mathbb{C}}, \mathbb{Z}/\ell)$$

are isomorphisms if $1/\ell \in k$. It is a consequence of the Riemann existence theorem [53, III.3.14] that there is an isomorphism

$$H^*(G_{\mathbb{C}}, \mathbb{Z}/\ell) \cong H^*(G(\mathbb{C}), \mathbb{Z}/\ell)$$

relating the étale cohomology groups $H^*(G_{\mathbb{C}},\mathbb{Z}/\ell)$ with the ordinary topological cohomology groups $H^*(G(\mathbb{C}),\mathbb{Z}/\ell)$ of the group $G(\mathbb{C})$ of complex points of G.

The topological cohomology groups $H^*(G(\mathbb{C}),\mathbb{Z}/\ell)$ are finite, so that the sheaves $\tilde{H}^*(G_{\mathscr{O}},\mathbb{Z}/\ell)$ on $(Sch|_{\mathscr{O}})_{et}$ are finite as well as constant, for all strict local hensel rings $\mathscr{O}$ with $1/\ell \in \mathscr{O}$.

It follows that the group-scheme G_S represents a sheaf on $(Sch|_S)_{et}$ which is cohomologically smooth and proper, provided that ℓ is distinct from the residue characteristics of S.

This means, in particular, that the sheaves $\tilde{H}^*(G_k,\mathbb{Z}/\ell)$ are constant and finite on $(Sch|_k)_{et}$ if k is a separably closed field with $1/\ell \in k$.

The formation of the isomorphisms

$$H^*(G_k,\mathbb{Z}/\ell) \xleftarrow{\cong} H^*(G_{\mathscr{O}^{sh}_{x,\mathbb{Z}}},\mathbb{Z}/\ell) \xrightarrow{\cong} H^*(G_{\mathbb{C}},\mathbb{Z}/\ell) \cong H^*(G(\mathbb{C}),\mathbb{Z}/\ell)$$

is a prototypical *base change* argument.

Theorem 8.37. *Suppose that $\mathscr{O}$ is a strict local hensel ring. Suppose that the simplicial presheaf Z on $(Sch|_{\mathscr{O}})_{et}$ is cohomologically smooth and proper with respect to ℓ-torsion sheaves. Then the cup product pairing*

$$\cup : H^*(X,\mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^*(Z,\mathbb{Z}/\ell) \to H^*(X \times Z,\mathbb{Z}/\ell)$$

is an isomorphism of graded rings, for all simplicial presheaves X on $(Sch|_{\mathscr{O}})_{et}$.

Proof. All tensor products will be defined over $\mathbb{Z}/\ell$.

Suppose that $\mathbb{Z}/\ell \to J$ is an injective resolution of $\mathbb{Z}/\ell$ in ℓ-torsion sheaves. The tensor product pairing

$$\hom(U,\mathbb{Z}/\ell) \otimes \hom(Z,J) \to \hom(U \times Z,J) \tag{8.17}$$

(of tricomplexes) induces a pairing

$$\pi(U,\mathbb{Z}/\ell[-p]) \otimes H^q(Z,\mathbb{Z}/\ell) \to H^{p+q}(U \times Z,\mathbb{Z}/\ell)$$

which natural in local weak equivalences $X \xleftarrow{\simeq} U$, and induces the cup product map

$$H^p(X,\mathbb{Z}/\ell) \otimes H^q(Z,\mathbb{Z}/\ell) \to H^{p+q}(X \times Z,\mathbb{Z}/\ell)$$

after taking colimits in $U \xrightarrow{\simeq} X$.

The pairing (8.17) is isomorphic to the pairing

$$\hom(U,\mathbb{Z}/\ell) \otimes \hom(*,\mathbf{Hom}(Z,J)) \to \hom(U,\mathbf{Hom}(Z,J)). \tag{8.18}$$

We will show that the induced maps

$$\begin{aligned} \bigoplus_{p+q=n} \pi_0 h_M(X,\mathbb{Z}/\ell[-p]) \otimes H^q \operatorname{Tot}(\hom(*,\mathbf{Hom}(Z,J))) \\ \to \pi_0 h_M(X,\operatorname{Tot}(\mathbf{Hom}(Z,J))[-n]) \end{aligned} \tag{8.19}$$

are isomorphisms. Then one finishes by using Lemma 8.14.

Consider the pairing of bicomplexes

$$\hom(U_p, \mathbb{Z}/\ell) \otimes \hom(*, \mathrm{Tot}(\mathbf{Hom}(Z,J))_q) \to \hom(U_p, \mathrm{Tot}(\mathbf{Hom}(Z,J))_q)$$

which is induced by the tricomplex map (8.18). Let $\mathrm{Tot}(\mathbf{Hom}(Z,J)) \to I$ be a Cartan-Eilenberg resolution, and form the diagram of tricomplexes

$$\begin{tikzcd}
\hom(U,\mathbb{Z}/\ell) \otimes \hom(*,\mathrm{Tot}(\mathbf{Hom}(Z,J)) \arrow[r] \arrow[d] \arrow[dr, dotted] & \hom(U,\mathrm{Tot}(\mathbf{Hom}(Z,J)) \arrow[d] \\
\hom(U,\mathbb{Z}/\ell) \otimes \hom(*,I) \arrow[r] & \hom(U,I)
\end{tikzcd}$$

The dotted arrow composite induces a map of spectral sequences which takes values in the Künneth spectral sequence for $H^*(U \times Z, \mathbb{Z}/\ell)$, with the map on E_2-terms given by the pairing

$$\pi(U,\mathbb{Z}/\ell[-p]) \otimes H^q(Z,\mathbb{Z}/\ell) \to H^p(U,\tilde{H}^q(Z,\mathbb{Z}/\ell)). \tag{8.20}$$

The assumption on Z means that the sheaves $\tilde{H}^*(Z,\mathbb{Z}/\ell)$ are finite and constant. It follows that the pairing (8.20) is an example of a cup product map

$$\pi(U,\mathbb{Z}/\ell[-p]) \otimes H^0(*,A) \xrightarrow{\cong} \pi(U,\mathbb{Z}/\ell[-p]) \otimes H^0(*,\tilde{A}) \to H^p(U,\tilde{A}), \tag{8.21}$$

where $\tilde{A} = \oplus_F \mathbb{Z}/\ell$ is the constant sheaf on a finite direct sum of copies of $\mathbb{Z}/\ell$. The map (8.21) can be identified up to isomorphism with the canonical map

$$\pi(U,A[-p]) \to H^p(U,A)$$

for coefficient sheaves A of this form, and the pairing (8.20) can therefore be identified up to isomorphism with the map

$$\pi(U,\tilde{H}^q(Z,\mathbb{Z}/\ell)[-p]) \to H^p(U,\tilde{H}^q(Z,\mathbb{Z}/\ell)) \tag{8.22}$$

from cocycles to cohomology classes.

The spectral sequence for the bicomplex

$$\hom(U,\mathbb{Z}/\ell) \otimes \hom(*,\mathrm{Tot}(\mathbf{Hom}(Z,J)))$$

collapses at the E_2-level for all U. The E_2-level comparison (8.20) is a canonical map from chain homotopy classes to cocycles, so for each $\alpha \in H^q(U,\tilde{H}^p\mathbf{Hom}(Z,J))$ there is a string of weak equivalences

$$U = U_0 \overset{\simeq}{\leftrightarrows} \dots \overset{\simeq}{\leftrightarrows} U_n = V$$

over X such that the image of α in $H^q(V,\mathrm{Tot}(\mathbf{Hom}(Z,J))_q)$ is in the image of the map

$$\pi(V,\mathbb{Z}/\ell[-q]) \otimes H^p(Z,\mathbb{Z}/\ell) \to H^q(V,\tilde{H}^p\mathbf{Hom}(Z,J)) \cong H^q(U,\tilde{H}^p\mathbf{Hom}(Z,J)).$$

It follows that the Künneth spectral sequence for $H^*(U \times Z,\mathbb{Z}/\ell)$ collapses at the E_2-level.

The map (8.19) is a map of filtered complexes, and is a colimit of the maps

$$\bigoplus_{p+q=n} \pi(U,\mathbb{Z}/\ell[-p]) \otimes H^q(Z,\mathbb{Z}/\ell) \to H^{p+q}\operatorname{Tot}(\hom(U,\operatorname{Tot}(\mathbf{Hom}(Z,J))).$$

The map on filtration quotients induced by (8.19) is an isomorphism, since it is a colimit of the canonical maps (8.22), and it follows that the map (8.19) is an isomorphism.

Corollary 8.38. *Suppose that $\mathscr{O}$ is a strict local hensel ring with $1/\ell \in \mathscr{O}$, and that $x : \mathrm{Sp}(k) \to \mathrm{Sp}(\mathscr{O})$ is a geometric point. Suppose that $G_{\mathbb{Z}}$ is an integral Chevalley form for a group scheme. Then the induced map*

$$x^* : H^*(BG_{\mathscr{O}},\mathbb{Z}/\ell) \to H^*(BG_k,\mathbb{Z}/\ell)$$

is an isomorphism.

Proof. The induced map

$$x^* : H^*(G_{\mathscr{O}}^{\times n},\mathbb{Z}/\ell) \to H^*(G_k^{\times n},\mathbb{Z}/\ell)$$

is isomorphic to the map

$$x^* : H^*(G_{\mathscr{O}},\mathbb{Z}/\ell)^{\otimes n} \to H^*(G_k,\mathbb{Z}/\ell)^{\otimes n}$$

by Theorem 8.37, and this map is an isomorphism since $G_{\mathbb{Z}}$ is cohomologically smooth and proper (Example 8.36). Complete the proof by comparing spectral sequences.

Corollary 8.39. *Suppose that $\mathscr{O}$ is a strict local hensel ring with $1/\ell \in \mathscr{O}$, where ℓ is a prime. Suppose that $G_{\mathbb{Z}}$ is an integral Chevalley form. Then there is an isomorphism*

$$H^*(BG_{\mathscr{O}},\mathbb{Z}/\ell) \cong H^*(BG(\mathbb{C}),\mathbb{Z}/\ell)$$

where $G(\mathbb{C})$ is the topological group of complex valued points of $G_{\mathbb{Z}}$.

Proof. There are geometric points

$$\mathscr{O} \to k \leftarrow \mathscr{O}_{x,\mathbb{Z}}^{sh} \to \mathbb{C}$$

for some point $x \in \mathrm{Sp}(\mathbb{Z})$. There are induced isomorphisms

$$\begin{aligned} H^*(BO_{n,\mathscr{O}},\mathbb{Z}/2) &\cong H^*(BO_{n,\bar{k}},\mathbb{Z}/2) \\ &\cong H^*(BO_{n,\mathscr{O}_{x,\mathbb{Z}}^{sh}},\mathbb{Z}/2) \\ &\cong H^*(BO_{n,\mathbb{C}},\mathbb{Z}/2) \end{aligned}$$

by Corollary 8.38. There is an isomorphism

$$H^*(BO_{n,\mathbb{C}},\mathbb{Z}/2) \cong H^*(BO_n(\mathbb{C}),\mathbb{Z}/2),$$

by Theorem 8.37 and a Riemann existence theorem argument.

Corollary 8.40. *Suppose that S is a scheme whose residue characteristics are distinct from ℓ, and that $G_{\mathbb{Z}}$ is an integral Chevalley form. Then the simplicial sheaf BG_S on $(Sch|_S)_{et}$ is cohomologically smooth and proper with respect to ℓ-torsion sheaves.*

Remark 8.41. Suppose that S is a scheme with residue characteristics distinct from a prime ℓ. Then one can prove the following:

1) The class of simplicial presheaves on $(Sch|_S)_{et})$ which are cohomologically smooth and proper with respect to ℓ-torsion sheaves is closed under finite products.
2) Suppose that X is a bisimplicial presheaf on $(Sch|_S)_{et}$ such that all vertical simplicial presheaves X_n are cohomologically smooth and proper with respect to ℓ-torsion sheaves. Then the diagonal object $d(X)$ is cohomologically smooth and proper with respect to ℓ-torsion sheaves.

Both statements are local with respect to the étale topology, so that one can presume that $S = \mathrm{Sp}(\mathscr{O})$, where $\mathscr{O}$ is a strict local hensel ring with $1/\ell \in \mathscr{O}$. Statement 1) is a direct consequence of Theorem 8.37, and the proof of statement 2) is a spectral sequence argument.

Theorem 8.42 below is due to Friedlander and Mislin. This result was, for a long time, the most general known case of the Friedlander-Milnor conjecture on the discrete cohomology of algebraic groups — see Chapter 3. The proof which is displayed here first appeared in [30].

Theorem 8.42. *Suppose that G is a reductive algebraic group which is defined over the finite field $\mathbb{F}_p$, and let ℓ be a prime such that $\ell \neq p$. Then the adjunction map*

$$\varepsilon : \Gamma^* BG(\overline{\mathbb{F}}_p) \to BG$$

of simplicial presheaves on the big étale site $(Sch|_{\overline{\mathbb{F}}_p})_{et}$ induces an isomorphism

$$\varepsilon^* : H^*_{et}(BG,\mathbb{Z}/\ell) \xrightarrow{\cong} H^*(BG(\overline{\mathbb{F}}_p),\mathbb{Z}/\ell).$$

Proof. The idea is to show that the presheaf $G/G(\overline{\mathbb{F}}_p)$ on $(Sch|_{\overline{\mathbb{F}}_p})_{et}$ has cohomology

$$H^n_{et}(G/G(\overline{\mathbb{F}}_p),\mathbb{Z}/\ell) \cong 0$$

for $n > 0$. In effect, the map ε can be identified in the homotopy category with the simplicial presheaf map

$$EG/G(\overline{\mathbb{F}}_p) \to BG$$

which consists of projections

$$(G \times \cdots \times G) \times G/G(\overline{\mathbb{F}}_p) \to G \times \cdots \times G \tag{8.23}$$

in the various simplicial degrees, and then one compares spectral sequences of the form

$$E_2^{p,q} = H^q(H^p(X_\bullet, \mathbb{Z}/\ell)) \Rightarrow H^{p+q}(X, \mathbb{Z}/\ell)$$

for simplicial presheaves X. The map in cohomology which is induced by the map (8.23) can be identified with the comparison

$$H^*(G, \mathbb{Z}/\ell)^{\otimes n} \to H^*(G, \mathbb{Z}/\ell)^{\otimes n} \otimes H^*(G/G(\overline{\mathbb{F}}_p), \mathbb{Z}/\ell)$$

by Theorem 8.37.

The object $G/G(\overline{\mathbb{F}}_p)$ is the colimit of the system

$$G/G(\mathbb{F}_p) \to G/G(\mathbb{F}_{p^2}) \to G/G(\mathbb{F}_{p^3}) \to \dots$$

Let $\phi : G \to G$ be the Frobenius automorphism. Then $G(\mathbb{F}_{p^n})$ is fixed by the automorphism ϕ^n and there is a short exact sequence

$$G(\mathbb{F}_{p^n}) \to G \xrightarrow{1/\phi^n} G,$$

of algebraic groups over $\overline{\mathbb{F}}_p$. Explicitly, there is an isomorphism

$$G/G(\mathbb{F}_{p^n}) \xrightarrow{\cong} G,\ x \mapsto x \cdot \phi^n(x)^{-1}$$

of varieties over $\overline{\mathbb{F}}_p$, called the *Lang isomorphism*.

Suppose that $\psi = \phi^n$ for some n. There is a commutative diagram

$$\begin{tikzcd}
 & G \arrow[r, "\cong"] & G/G(\mathbb{F}_{p^{nd}}) \\
G \arrow[ur, "1/\psi^d"] \arrow[dr, "1/\psi"'] & & \\
 & G \arrow[uu, "\gamma"] \arrow[r, "\cong"'] & G/G(\mathbb{F}_{p^n}) \arrow[uu]
\end{tikzcd}$$

where

$$\gamma(x) = x \cdot \psi(x) \cdot \psi^2(x) \cdot \dots \cdot \psi^{d-1}(x).$$

It follows that the induced map $\gamma^* : H^*_{et}(G, \mathbb{Z}/\ell) \to H^*_{et}(G, \mathbb{Z}/\ell)$ satisfies

$$\gamma^*(x) = x + \psi^*(x) + \dots + (\psi^*)^{d-1}(x) + \text{decomposables}$$

with respect to the Hopf algebra structure on $H^*_{et}(G, \mathbb{Z}/\ell)$.

The map ϕ^* is an automorphism of the finite ℓ-torsion group $H^k_{et}(G, \mathbb{Z}/\ell)$, as is ψ^*, so that $(\psi^*)^N = 1$ for some N. Thus, if $d = N\ell$ then $\gamma^*(H^k_{et}(G, \mathbb{Z}/\ell))$ is decomposable. By induction on k, the system of pro-groups

$$\{H^k_{et}(G/G(\mathbb{F}_{p^n}),\mathbb{Z}/\ell)\}$$

is therefore pro-trivial, so that

$$H^k(G/G(\overline{\mathbb{F}}_p),\mathbb{Z}/\ell)=0$$

for $k \geq 0$, by a Milnor exact sequence argument.

Remark 8.43. Suppose that X is the colimit of a system

$$X_1 \to X_2 \to X_3 \to \dots$$

of simplicial presheaf maps and A is a presheaf of abelian groups, and take an injective fibrant model $j : K(A,n) \to GK(A,n)$. We can assume, without loss of generality, that all of the maps $X_k \to X_{k+1}$ are cofibrations. Then $\mathbf{hom}(X, GK(A,n))$ is the inverse limit of a tower of fibrations

$$\mathbf{hom}(X_1, GK(A,n)) \leftarrow \mathbf{hom}(X_2, GK(A,n)) \leftarrow \dots$$

and the first Milnor exact sequence [8, IX.3.1] for the tower has the form

$$0 \to \varprojlim_k{}^1 H^{n-1}(X_k,A) \to H^n(X,A) \to \varprojlim_k H^n(X_k,A) \to 0 \tag{8.24}$$

if $n \geq 1$. This is the device which is used at the end of the proof of Theorem 8.42.

Theorem 8.37 and its corollaries depend on specific properties of étale sites. It is a much more general phenomenon that the cohomology of the constant simplicial presheaf Γ^*X for a simplicial set X (aka. Grothendieck's mixed cohomology: Example 8.27) has a particularly nice cup product decomposition, provided that X is homologically of finite type.

Lemma 8.44. *Suppose that ℓ is a prime number, and that X is a simplicial set such that all homology groups $H_i(X,\mathbb{Z}/\ell)$ are finite. Suppose that Y is a simplicial presheaf and that A is an ℓ-torsion abelian sheaf on a small site $\mathscr{C}$. Then the cup product pairings*

$$H^p(X,\mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^q(Y,A) \to H^{p+q}(\Gamma^*X \times Y, A)$$

which are induced by the pairing $\mathbb{Z}/\ell \otimes A \cong A$ define an isomorphism

$$H^*(X,\mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^*(Y,A) \xrightarrow{\cong} H^*(\Gamma^*X \times Y, A)$$

of graded ℓ-torsion modules.

Proof. Write $H_p(X) = H_p(X,\mathbb{Z}/\ell)$ and suppose that all tensor products are over $\mathbb{Z}/\ell$ in the following.

Suppose that D is an ordinary cochain complex of ℓ-torsion modules. The tensor product pairing

$$\hom(X_q, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} \hom(*, D^p) \to \hom(X_q \times *, \mathbb{Z}/\ell \otimes D^p)$$

defines a bicomplex morphism

$$\hom(X, \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} D \to \hom(X, D).$$

A comparison of spectral sequences shows that this bicomplex morphism induces a weak equivalence of associated total complexes, since all homology groups $H_n(X, \mathbb{Z}/\ell)$ are finite. In effect, the map on E_2-terms is the canonical pairing

$$H^q(X, \mathbb{Z}/\ell) \otimes H^p(D) \to H^q(X, H^p(D)).$$

The isomorphic pairing

$$\hom(H_q(X), \mathbb{Z}/\ell) \otimes H^p(D) \to \hom(H_q(X), H^p(D))$$

takes the pair (f, α) to the morphism $x \mapsto f(x) \cdot \alpha$, and this map is an isomorphism since the groups $H_q(X)$ are finite.

There are isomorphisms

$$\hom(\Gamma^* X \times Y, J) \cong \hom(\Gamma^* X, \mathbf{Hom}(Y, J)) \cong \hom(X, \hom(Y, J))$$

and we use $D = \mathrm{Tot}(hom(Y, J))$ as in the previous paragraph to finish the proof.

Lemma 8.45. *Suppose that ℓ is a prime number, and suppose that $\mathscr{C}$ is a site which has the property that the canonical map $F \to \Gamma_* \Gamma^* F$ is a bijection for all finite sets F. Suppose that Y is a simplicial set such that all groups $H_k(Y, \mathbb{Z}/\ell)$ are finite. Suppose that X is a simplicial presheaf on $\mathscr{C}$. Then the Künneth spectral sequence*

$$E_2^{p,q} = H^p(X, \tilde{H}^q(\Gamma^* Y, \mathbb{Z}/\ell)) \Rightarrow H^{p+q}(X \times \Gamma^* Y, \mathbb{Z}/\ell)$$

collapses at the E_2-level.

All étale sites for connected schemes S satisfy the condition on the site $\mathscr{C}$ of Lemma 8.45. In that case, the constant sheaf $\Gamma^* F$ on a finite set F is represented by the scheme $\bigsqcup_F S$.

Proof. We use the machinery which was developed for the proof of Theorem 8.37.

Suppose that $\mathbb{Z}/\ell \to J$ is an injective resolution in ℓ-torsion sheaves, and that the map $U \to X$ is a local weak equivalence. Let

$$\mathrm{Tot}(\mathbf{Hom}(\Gamma^* Y, J)) \to I$$

be a Cartan-Eilenberg resolution, and consider the bicomplex morphisms

$$\begin{tikzcd}
\hom(U, \mathbb{Z}/\ell) \otimes \hom(Y, \mathbb{Z}/\ell) \arrow[r] \arrow[dr, dotted] & \hom(U, \mathbb{Z}/\ell) \otimes \hom(*, \mathrm{Tot}(\mathbf{Hom}(\Gamma^* Y, J))) \arrow[d] \\
& \hom(U, \mathrm{Tot}(I))
\end{tikzcd}$$

The horizontal morphism in the diagram is induced by the composite cochain complex map

$$\hom(Y,\mathbb{Z}/\ell) \xrightarrow{\Gamma^*} \hom(\Gamma^*Y,\mathbb{Z}/\ell).$$

Comparing spectral sequences along the dotted arrow composite gives an E_2-level map

$$\pi(U,\mathbb{Z}/\ell[-p]) \otimes H^q(Y,\mathbb{Z}/\ell) \to H^p(X,\tilde{H}^q(\Gamma^*Y,\mathbb{Z}/\ell)),$$

and this map may be identified up to isomorphism with the canonical morphism

$$\pi(U,\tilde{H}^q(\Gamma^*Y,\mathbb{Z}/\ell)[-p]) \to H^p(X,\tilde{H}^q(\Gamma^*Y,\mathbb{Z}/\ell))$$

by the assumptions on the simplicial set Y and the site $\mathscr{C}$. In effect, we use Lemma 8.44 to show that $\tilde{H}^*(\Gamma^*Y,\mathbb{Z}/\ell)$ is the sheaf associated to the presheaf

$$U \mapsto H^*(Y,\mathbb{Z}/\ell) \otimes H^*(U,\mathbb{Z}/\ell).$$

It follows that there is an isomorphism

$$\tilde{H}^*(\Gamma^*Y,\mathbb{Z}/\ell) \cong \Gamma^*(H^*(Y,\mathbb{Z}/\ell))$$

since $\mathbb{Z}/\ell \to \Gamma_*\Gamma^*\mathbb{Z}/\ell$ is an isomorphism of groups.

Finish the proof as in the argument for Theorem 8.37.

We finish this section with a calculation. Suppose that S is a scheme, and let $O_{n,S} = S \times O_n$ be the standard orthogonal group over S. In affine sections, the group-scheme O_n over $\mathbb{Z}$ is the e of $(n \times n)$-matrices A such that $A \cdot A^t = A^t \cdot A = I_n$. It is also the group of automorphisms of the trivial symmetric form of rank n. Then we have the following result of [35]:

Theorem 8.46. *Suppose that S is a connected scheme such that 2 is invertible in all residue fields, and let X be a simplicial presheaf on the site $(Sch|_S)_{et}$. Then there is an isomorphism*

$$H^*(X \times BO_{n,S},\mathbb{Z}/2) \cong H^*(X,\mathbb{Z}/2)[HW_1,HW_2,\dots,HW_n],$$

of rings, where the degree of the polynomial generator HW_i is i.

The polynomial generators HW_i of Theorem 8.46 are the *universal Hasse-Witt classes* [34].

Proof. The inclusion $i : \mathbb{Z}/2^{\times n} \subset O_{n,\mathbb{Z}}$ induces a simplicial sheaf map

$$i : \Gamma^*B\mathbb{Z}/2^{\times n} \to BO_{n,S}.$$

The symmetric group Σ_n acts on $\mathbb{Z}/2^{\times n}$ by permuting factors.

Suppose that $p : \mathscr{O} \to S$ is a scheme homomorphism where $\mathscr{O}$ is a strict local hensel ring.

A base change argument (Corollary 8.40) says that there is an isomorphism

$$H^*(BO_{n,\mathscr{O}},\mathbb{Z}/2) \xrightarrow{\cong} H^*(B\mathbb{Z}/2^{\times n},\mathbb{Z}/2)^{\Sigma_n} \cong \mathbb{Z}/2[\sigma_1,\dots,\sigma_n],$$

where σ_i is the i^{th} elementary symmetric polynomial in the degree 1 generators $x_1,\dots,x_n$ for the space $B\mathbb{Z}/2^{\times n}$. It follows that the map $i : \Gamma^*B\mathbb{Z}/2^{\times n} \to BO_{n,s}$ induces an isomorphism

$$\tilde{H}^*(BO_{n,S},\mathbb{Z}/2) \xrightarrow{\cong} \tilde{H}^*(\Gamma^*B\mathbb{Z}/2^{\times n},\mathbb{Z}/2)^{\Sigma_n}.$$

The map i induces a comparison

$$H^p(X,\tilde{H}^q(BO_{n,S},\mathbb{Z}/2)) \to H^p(X,\tilde{H}^q(\Gamma^*B\mathbb{Z}/2^{\times n},\mathbb{Z}/2))$$

of Künneth spectral sequences at the E_2-level. This map is induced by a split monomorphism of coefficient sheaves by the previous paragraph, and is therefore a monomorphism. The Künneth spectral sequence for $H^*(\Gamma^*B\mathbb{Z}/2^{\times n},\mathbb{Z}/2)$ collapses at the E_2-level by Lemma 8.44 and the connectedness assumption for S. It follows that the Künneth spectral sequence for $H^*(BO_{n,S},\mathbb{Z}/2)$ collapses at the E_2-level. It also follows that the map

$$i^* : H^*(BO_{n,S},\mathbb{Z}/2) \to H^*(\Gamma^*B\mathbb{Z}/2^{\times n},\mathbb{Z}/2)$$

is a monomorphism.

The filtration on the group

$$H^{p+q}(X\times\Gamma^*B\mathbb{Z}/2^{\times n},\mathbb{Z}/2)$$

arising from the Künneth spectral sequence is Σ_n-equivariant, and restricts to a filtration on

$$H^{p+q}(X\times\Gamma^*B\mathbb{Z}/2^{\times n},\mathbb{Z}/2)^{\Sigma_n}$$

with filtration quotients

$$H^p(X,\mathbb{Z}/2)\otimes H^q(B\mathbb{Z}/2^{\times n},\mathbb{Z}/2)^{\Sigma_n}.$$

The map

$$H^{p+q}(X\times BO_{n,S},\mathbb{Z}/2) \to H^{p+q}(X\times\Gamma^*B\mathbb{Z}/2^{\times m},\mathbb{Z}/2)$$

induces a map

$$H^{p+q}(X\times BO_{n,S},\mathbb{Z}/2) \to H^{p+q}(X\times\Gamma^*B\mathbb{Z}/2^{\times n},\mathbb{Z}/2)^{\Sigma_n}$$

of filtered groups which is an isomorphism on filtration quotients.

Chapter 9
Non-abelian cohomology

9.1 Torsors for groups

Suppose that G is a sheaf of groups. A *G-torsor* is traditionally defined to be a sheaf X with a free G-action such that $X/G \cong *$ in the sheaf category.

The simplicial sheaf $EG \times_G X$ is the nerve of a sheaf of groupoids, which is given in each section by the translation category for the action of $G(U)$ on $X(U)$ — see Example 1.7. It follows that all sheaves of higher homotopy groups for $EG \times_G X$ vanish. The requirement that the action $G \times X \to X$ is free means that the isotropy subgroups of G for the action are trivial in all sections, which is equivalent to requiring that all sheaves of fundamental groups for the Borel construction $EG \times_G X$ are trivial. Finally, there is an isomorphism of sheaves

$$\tilde{\pi}_0(EG \times_G X) \cong X/G.$$

These observations together imply the following:

Lemma 9.1. *A sheaf X with G-action is a G-torsor if and only if the simplicial sheaf map $EG \times_G X \to *$ is a local weak equivalence.*

Example 9.2. If G is a sheaf of groups, then $EG = EG \times_G G$ is contractible in each section, so that the map

$$EG \times_G G \to *$$

is a local weak equivalence, and G is a G-torsor. This object is often called the *trivial G-torsor*.

Example 9.3. Suppose that L/k is a finite Galois extension of fields with Galois group G. Let $C(L)$ be the Čech resolution for the étale covering $\mathrm{Sp}(L) \to \mathrm{Sp}(k)$, as in Example 4.18. Then there is an isomorphism of simplicial schemes

$$C(L) \cong EG \times_G \mathrm{Sp}(L),$$

and the simplicial presheaf map $C(L) \to *$ on $Sch|_k$ is a local weak equivalence for the étale topology. The k-scheme $\mathrm{Sp}(L)$ represents a G-torsor for all of the standard étale sites associated with the field k.

The category $G-\mathbf{tors}$ of G-torsors is the category whose objects are all G-torsors and whose maps are all G-equivariant maps between them.

Lemma 9.4. *Suppose that G is a sheaf of groups. Then the category $G-\mathbf{tors}$ of G-torsors is a groupoid.*

Proof. If $f : X \to Y$ is a map of G-torsors, then f is induced as a map of fibres by the comparison of local fibrations

$$\begin{tikzcd} EG\times_G X \arrow[rr, "\simeq"] \arrow[dr] & & EG\times_G Y \arrow[dl] \\ & BG & \end{tikzcd}$$

The map $f : X \to Y$ of fibres is a weak equivalence of constant simplicial sheaves by Lemma 5.16 and properness of the injective model structure for simplicial sheaves, and is therefore an isomorphism.

Remark 9.5. Suppose that X is a G-torsor, and that the canonical map $X \to *$ has a (global) section $\sigma : * \to X$. Then σ extends, by multiplication, (also uniquely) to a G-equivariant map

$$\sigma_* : G \to X,$$

with $\sigma_*(g) = g \cdot \sigma_U$ for $g \in G(U)$. This map is an isomorphism of torsors, so that X is trivial with trivializing isomorphism σ_*. Conversely, if $\tau : G \to X$ is a map of torsors, then X has a global section $\tau(e)$. Thus a G-torsor X is trivial in the sense that it is isomorphic to G if and only if it has a global section.

Example 9.6. Suppose that X is a topological space. The category of sheaves on $op|_X$ can be identified up to equivalence with the slice category Top/X of spaces $Y \to X$ fibred over X.

If G is a topological group, then G represents the sheaf $G \times X \to X$ given by projection. A sheaf with G-action consists of a map $Y \to X$ together with a G-action $G \times Y \to Y$ such that the map $Y \to X$ is G-equivariant for the trivial G-action on X. Such a thing is a G-torsor if the action $G \times Y \to Y$ is free and the map $Y/G \to X$ is an isomorphism. The latter implies that X has an open covering $i : U \subset X$ such that there are liftings

$$\begin{tikzcd} & Y \arrow[d] \\ U \arrow[ur, "\sigma"] \arrow[r, "i"'] & X \end{tikzcd}$$

Torsors are stable under pullback along continuous maps, and the map $U \times_X Y \to U$ is a G-torsor over U. The map σ induces a global section σ_* of this map, so that the pulled back torsor is trivial, and there is a commutative diagram

$$\begin{tikzcd} G \times U \arrow[r, "\cong"] \arrow[dr, "pr"'] & U \times_X Y \arrow[d] \\ & U \end{tikzcd}$$

where the displayed isomorphism is G-equivariant. It follows that a G-torsor over X is a principal G-bundle over X, and conversely.

Example 9.7. Suppose that U is an object of a small site $\mathscr{C}$. Composition with the forgetful functor $q : \mathscr{C}/U \to \mathscr{C}$ defines a restriction functor

$$\mathbf{Shv}(\mathscr{C}) \to \mathbf{Shv}(\mathscr{C}/U),$$

written $F \mapsto F|_U$. The restriction functor is exact, and therefore takes G-torsors to $G|_U$-torsors. The global sections of $F|_U$ coincide with the elements of the set $F(U)$, so that a G-torsor X trivializes over U if and only if $X(U) \neq \emptyset$, or if and only if there is a diagram

$$\begin{tikzcd} & X \arrow[d] \\ U \arrow[ur] \arrow[r] & * \end{tikzcd}$$

The map $X \to *$ is a local epimorphism, so there is a covering family $U_\alpha \to *$ (ie. such that $\bigsqcup U_\alpha \to *$ is a local epimorphism) with $X(U_\alpha) \neq \emptyset$. In other words, every torsor trivializes over some covering family of the point $*$.

Suppose that the picture

$$* \xleftarrow{\simeq} Y \xrightarrow{\alpha} BG$$

is an object of the cocycle category $h(*, BG)$ in simplicial presheaves, and form the pullback

$$\begin{tikzcd} \mathrm{pb}(Y) \arrow[r] \arrow[d] & Y \arrow[d, "\alpha"] \\ EG \arrow[r, "\pi"'] & BG \end{tikzcd}$$

where $EG = B(G/*) = EG \times_G G$ and $\pi : EG \to BG$ is the canonical map. Then $\mathrm{pb}(Y)$ inherits a G-action from the G-action on EG, and the map

$$EG \times_G \mathrm{pb}(Y) \to Y \tag{9.1}$$

is a sectionwise weak equivalence (this is a consequence of Lemma 9.9 below). Also, the square is homotopy cartesian in sections, so there is a local weak equivalence

$$G|_U \to \mathrm{pb}(Y)|_U$$

where $Y(U) \neq \emptyset$. It follows that the canonical map $\mathrm{pb}(Y) \to \tilde{\pi}_0\,\mathrm{pb}(Y)$ is a G-equivariant local weak equivalence, and hence that the maps

$$EG \times_G \tilde{\pi}_0\,\mathrm{pb}(Y) \leftarrow EG \times_G \mathrm{pb}(Y) \to Y \simeq *$$

are natural local weak equivalences. In particular, the G-sheaf $\tilde{\pi}_0\,\mathrm{pb}(Y)$ is a G-torsor.

We therefore have a functor

$$h(*,BG) \to G - \mathbf{tors}$$

defined by sending $* \xleftarrow{\simeq} Y \to BG$ to the object $\tilde{\pi}_0\,\mathrm{pb}(Y)$. The Borel construction defines a functor

$$G - \mathbf{tors} \to h(*,BG)$$

in which the G-torsor X is sent to the (canonical) cocycle

$$* \xleftarrow{\simeq} EG \times_G X \to BG.$$

It is elementary to check (see also the proof of Theorem 9.14 below) that these two functors induce a bijection

$$\pi_0 h(*,BG) \cong \pi_0(G - \mathbf{tors}).$$

The set $\pi_0(G - \mathbf{tors})$ is isomorphism classes of G-torsors, and we know from Theorem 5.34 that there is an isomorphism

$$\pi_0 h(*,BG) \cong [*,BG].$$

The non-abelian invariant $H^1(\mathscr{C},G)$ is traditionally defined to be the collection of isomorphism classes of G-torsors. We have therefore proved the following:

Theorem 9.8. *Suppose that G is a sheaf of groups on a small Grothendieck site $\mathscr{C}$. Then there is a bijection*

$$[*,BG] \cong H^1(\mathscr{C},G).$$

Theorem 9.8 was first proved, by a different method, in [34].

Here's the missing lemma:

Lemma 9.9. *Suppose that I is a small category and that $p : X \to I$ is a simplicial set map. Let the pullback diagrams*

$$\begin{tikzcd} \mathrm{pb}(X)(i) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B(I/i) \arrow[r] & BI \end{tikzcd}$$

define the I-diagram $i \mapsto \mathrm{pb}(X)(i)$. Then the resulting map

$$\omega : \underrightarrow{\mathrm{holim}}_{i\in I}\ \mathrm{pb}(X)(i) \to X$$

is a weak equivalence.

Proof. The simplicial set

$$\underrightarrow{\mathrm{holim}}_{i\in I}\ \mathrm{pb}(X)(i)$$

is the diagonal of a bisimplicial set whose (n,m)-bisimplices are pairs

$$(x, i_0 \to \cdots \to i_n \to j_0 \to \cdots \to j_m)$$

where $x \in X_n$, the morphisms are in I, and $p(x)$ is the string

$$i_0 \to \cdots \to i_n.$$

The map

$$\omega : \underrightarrow{\mathrm{holim}}_{i\in I}\ \mathrm{pb}(X)(i) \to X$$

takes such an (n,m)-bisimplex to $x \in X_n$. The fibre of ω over x in vertical degree n can be identified with the simplicial set $B(i_n/I)$, which is contractible.

Example 9.10. Suppose that k is a field. Let $\mathscr{C}$ be the étale site $et|_k$ for k, and identify the orthogonal group $O_n = O_{n,k}$ with a sheaf of groups on this site. The non-abelian cohomology object $H^1_{et}(k, O_n)$ coincides with the set of isomorphism classes of non-degenerate symmetric bilinear forms over k of rank n. Thus, every such form q determines a morphism $* \to BO_n$ in the simplicial (pre)sheaf homotopy category on $Sch|_k$, and this morphism determines the form q up to isomorphism.

Suppose that k is a field such that $char(k) \neq 2$. Then, by Theorem 8.46, there is a ring isomorphism

$$H^*_{et}(BO_{n,k}, \mathbb{Z}/2) \cong H^*_{et}(k, \mathbb{Z}/2)[HW_1, \dots, HW_n]$$

where the polynomial generator HW_i has degree i. The generator HW_i is characterized by mapping to the i^{th} elementary symmetric polynomial $\sigma_i(x_1, \dots, x_n)$ under the map

$$H^*(BO_{n,k}, \mathbb{Z}/2) \to H^*(\Gamma^* B\mathbb{Z}/2^{\times n}, \mathbb{Z}/2) \cong H^*_{et}(k, \mathbb{Z}/2)[x_1, \dots, x_n].$$

Every symmetric bilinear form α determines a map $\alpha : * \to BO_n$ in the simplicial presheaf homotopy category, and therefore induces a map

$$\alpha^* : H^*_{et}(BO_n, \mathbb{Z}/2) \to H^*_{et}(k, \mathbb{Z}/2),$$

and $HW_i(\alpha) = \alpha^*(HW_i)$ is the i^{th} *Hasse-Witt class* of α.

One can show that $HW_1(\alpha)$ is the pullback of the determinant $BO_n \to B\mathbb{Z}/2$, and $HW_2(\alpha)$ is the classical Hasse-Witt invariant of α.

The Steenrod algebra is used to calculate the relation between Hasse-Witt and Stiefel-Whitney classes for Galois representations. This calculation uses the Wu formulas for the action of the Steenrod algebra on elementary symmetric polynomials — see [34], [35]. **We shall discuss Steenrod operations in more detail later**

Example 9.11. Suppose that S is a scheme. The general linear group Gl_n represents a sheaf of groups on the étale site $(Sch|_S)_{et}$ and the sheaf of groups $\mathbb{G}_m$ can be identified with the centre of Gl_n via the diagonal imbedding $\mathbb{G}_m \to GL_n$. There is a short exact sequence

$$e \to \mathbb{G}_m \to Gl_n \xrightarrow{p} PGl_n \to e$$

of sheaves of groups on $(Sch|_S)_{et}$. The projective general linear group PGl_n can be identified with the group scheme of automorphisms $Aut(M_n)$ of the scheme of $(n \times n)$-matrices M_n, and the homomorphism p takes an invertible matrix A to the automorphism defined by conjugation by A.

Since $\mathbb{G}_m$ is a central subgroup of Gl_n, there is an induced action

$$B\mathbb{G}_m \times BGl_n \to BGl_n$$

of the simplicial abelian group $B\mathbb{G}_m$ on the simplicial sheaf BGl_n, and there is an induced sectionwise (hence local) fibre sequence associated to the sequence of bisimplicial objects

$$BGl_n \to EB\mathbb{G}_m \times_{B\mathbb{G}_m} BGl_n \xrightarrow{\pi} BB\mathbb{G}_m \simeq K(\mathbb{G}_m, 2)$$

after taking diagonals.

In effect, if $A \times X \to X$ is an action of a connected simplicial abelian group A on a connected simplicial set X then all sequences

$$X \to A^{\times n} \times X \to A^{\times n}$$

are fibre sequences of connected simplicial sets, so that the sequence

$$X \to EA \times_A X \to BA$$

of bisimplicial set maps induces a fibre sequence of simplicial sets after taking diagonals, by a theorem of Bousfield and Friedlander [24, IV.4.9].

The $B\mathbb{G}_m$ action on BGl_n is free, so there is a local weak equivalence

$$EB\mathbb{G}_m \times_{B\mathbb{G}_m} BGl_n \xrightarrow{\simeq} BPGl_n.$$

It follows that the map π induces a function

$$H^1_{et}(S, PGl_n) = [*, BPGl_n] \xrightarrow{d := \pi_*} [*, K(\mathbb{G}_m, 2)] = H^2_{et}(S, \mathbb{G}_m).$$

The set $H^1_{et}(S, Gl_n)$ is isomorphism classes of vector bundles over S of rank n, and the set $H^1_{et}(S, PGl_n)$ is the set of isomorphisms classes of rank n^2 Azumaya algebras. The map

$$p_* : [*, BGl_n] \to [*, BPGl_n]$$

takes a vector bundle E to the Azumaya algebra $\mathbf{End}(E)$ which is defined by the sheaf of endomorphisms of the S-module E.

Recall that the *Brauer group* $Br(S)$ is the abelian group of similarity classes of Azumaya algebras over S: the Azumaya algebras A and B are similar if there are vector bundles E and F such that there are isomorphisms

$$A \otimes \mathbf{End}(E) \cong B \otimes \mathbf{End}(F).$$

The group stucture on $Br(S)$ is induced by tensor product of Azumaya algebras.

More explicitly, tensor product of modules induces a comparison of exact sequences

$$\begin{tikzcd}
e \arrow[r] & \mathbb{G}_m \times \mathbb{G}_m \arrow[r] \arrow[d, "+"'] & Gl_n \times Gl_m \arrow[r, "p \times p"] \arrow[d, "\otimes"'] & PGl_n \times PGl_m \arrow[r] \arrow[d, "\otimes"] & e \\
e \arrow[r] & \mathbb{G}_m \arrow[r] & Gl_{nm} \arrow[r, "p"'] & PGl_{nm} \arrow[r] & e
\end{tikzcd}$$

where $+$ is the group structure on $\mathbb{G}_m$, and the induced map

$$\otimes : [*, BPGl_n] \times [*, BPGl_m] \to [*, BPGl_{nm}]$$

defines the tensor product of Azumaya algebras. We also have induced commutative diagrams

$$\begin{tikzcd}
{[*, BPGl_n] \times [*, BPGl_m]} \arrow[r, "d \times d"] \arrow[d, "\otimes"'] & H^2_{et}(S, \mathbb{G}_m) \times H^2_{et}(S, \mathbb{G}_m) \arrow[d, "+"] \\
{[*, BPGl_{nm}]} \arrow[r, "d"'] & H^2_{et}(S, \mathbb{G}_m)
\end{tikzcd}$$

in which the displayed pairing on $H^2(S, \mathbb{G}_m)$ is the abelian group addition. It follows that the collection of morphisms

$$d : [*, BPGl_n] \to H^2_{et}(S, \mathbb{G}_m)$$

define a group homomorphism

$$d : Br(S) \to H^2_{et}(S, \mathbb{G}_m).$$

This homomorphism d is a monomorphism: if $d(A) = 0$ for some Azumaya algebra A, then there is an isomorphism $A \cong \mathbf{End}(E)$ for some vector bundle E by the exactness of the sequence

$$[*, BGl_n] \xrightarrow{p_*} [*, BPGl_n] \xrightarrow{d} H^2_{et}(S, \mathbb{G}_m),$$

so that A represents 0 in the Brauer group.

Finally, if S is connected then the Brauer group $Br(S)$ consists of torsion elements. As in [53, IV.2.7], this follows from the existence of the diagram of short exact sequences of sheaves of groups

$$\begin{tikzcd}
 & e \arrow[d] & e \arrow[d] & & \\
e \arrow[r] & \mu_n \arrow[r] \arrow[d] & \mathbb{G}_m \arrow[r, "\times n"] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, "1"] & e \\
e \arrow[r] & Sl_n \arrow[r] \arrow[d] & Gl_n \arrow[r, "det"'] \arrow[d] & \mathbb{G}_m \arrow[r] & e \\
 & PGl_n \arrow[r, "1"'] \arrow[d] & PGl_n \arrow[d] & & \\
 & e & e & &
\end{tikzcd}$$

on the étale site $(Sch|_S)_{et}$, where μ_n is the subgroup of n-torsion elements in $\mathbb{G}_m$. The vertical sequence on the left is a central extension, so that there is a map $d : [*, BPGl_n] \to H^*_{et}(S, \mu_n)$ which fits into a commutative diagram

$$\begin{tikzcd}
{[*, BPGl_n]} \arrow[r, "d"] \arrow[d, "1"'] & H^2_{et}(S, \mu_n) \arrow[d] \\
{[*, BPGl_n]} \arrow[r, "d"'] & H^2_{et}(S, \mathbb{G}_m)
\end{tikzcd}$$

and $H^2_{et}(S, \mu_n)$ is an n-torsion abelian group. It follows that the Brauer group $Br(S)$ consists of torsion elements if the scheme S has finitely many components.

The assertion that there is monomorphism

$$d : Br(S) \to H^2_{et}(S, \mathbb{G}_m)_{tors}$$

into the torsion part of $H^2_{et}(S, \mathbb{G}_m)$ is a well known theorem of étale cohomology theory [53, IV.2.5]. The distinctive feature of the the present development is the use of easily defined fibre sequences of simplicial sheaves to produce the map d.

9.2 Torsors for groupoids

Suppose that I is a small category. A functor $X : I \to \mathbf{Set}$ consists of sets $X(i)$, $i \in \mathrm{Ob}(I)$ and functions $\alpha_* : X(i) \to X(j)$ for $\alpha : i \to j$ in $\mathrm{Mor}(I)$ such that $\alpha_*\beta_* = (\alpha \cdot \beta)_*$ for all composable pairs of morphisms in I and $(1_i)_* = 1_{X(i)}$ for all objects i of I.

The sets $X(i)$ can be collected together to give a set

$$\pi : X = \bigsqcup_{i \in \mathrm{Ob}(I)} X(i) \to \bigsqcup_{i \in \mathrm{Ob}(I)} = \mathrm{Ob}(I)$$

and the assignments $\alpha \mapsto \alpha_*$ can be collectively rewritten as a commutative diagram

$$\begin{tikzcd} X \times_{\pi,s} \mathrm{Mor}(I) \arrow[r, "m"] \arrow[d, "pr"'] & X \arrow[d, "\pi"] \\ \mathrm{Mor}(I) \arrow[r, "t"'] & \mathrm{Ob}(I) \end{tikzcd} \tag{9.2}$$

where $s,t : \mathrm{Mor}(I) \to \mathrm{Ob}(I)$ are the source and target maps, respectively, and

$$\begin{tikzcd} X \times_{\pi,s} \mathrm{Mor}(I) \arrow[r, "pr"] \arrow[d] & \mathrm{Mor}(I) \arrow[d, "s"] \\ X \arrow[r, "\pi"'] & \mathrm{Ob}(I) \end{tikzcd}$$

is a pullback. Then the notation is awkward, but the composition laws for the functor X translate into the commutativity of the diagrams

$$\begin{tikzcd} X \times_{\pi,s} \mathrm{Mor}(I) \times_{t,s} \mathrm{Mor}(I) \arrow[r, "1 \times m_I"] \arrow[d, "m \times 1"'] & X \times_{\pi,s} \mathrm{Mor}(I) \arrow[d, "m"] \\ X \times_{\pi,s} \mathrm{Mor}(I) \arrow[r, "m"'] & X \end{tikzcd} \tag{9.3}$$

and

$$\begin{tikzcd} X \arrow[r, "e_*"] \arrow[dr, "1"'] & X \times_{\pi,s} \mathrm{Mor}(I) \arrow[d, "m"] \\ & X \end{tikzcd} \tag{9.4}$$

Here, m_I is the composition law of the category I, and the map e_* is uniquely determined by the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "1"'] & \mathrm{Ob}(I) \arrow[r, "e"] & \mathrm{Mor}(I) \arrow[d, "s"] \\ X \arrow[rr, "\pi"'] & & \mathrm{Ob}(I) \end{tikzcd}$$

where the map e picks out the identity morphisms of I.

Thus, a functor $X : I \to \mathbf{Set}$ consists of a function $\pi : X \to \mathrm{Ob}(I)$ together with an action $m : X \times_{\pi,s} \mathrm{Mor}(I) \to X$ making the diagram (9.2) commute, and such that the diagrams (9.3) and (9.4) also commute. This is the *internal description* of a functor, which description can be used to define functors on category objects.

Specifically, suppose that G is a sheaf of groupoids on a site $\mathscr{C}$. Then a *sheaf-valued functor* X on G, or more commonly a *G-functor*, consists of a sheaf map $\pi : X \to \mathrm{Ob}(G)$, together with an action morphism $m : X \times_{\pi,s} \mathrm{Mor}(G) \to X$ in sheaves such that the diagrams corresponding to (9.2), (9.3) and (9.4) commute in the sheaf category.

Alternatively, X consists of set-valued functors

$$X(U) : G(U) \to \mathbf{Sets}$$

with $x \mapsto X(U)_x$ for $x \in \mathrm{Ob}(G(U))$, together with functions

$$\phi^* : X(U)_x \to X(V)_{\phi^*(x)}$$

for each $\phi : V \to U$ in $\mathscr{C}$, such that the assignment

$$U \mapsto X(U) = \bigsqcup_{x \in \mathrm{Ob}(G(U))} X(U)_x, \; U \in \mathscr{C},$$

defines a sheaf and the diagrams

$$\begin{tikzcd} X(U)_x \arrow[r, "\alpha_*"] \arrow[d, "\phi^*"'] & X(U)_y \arrow[d, "\phi^*"] \\ X(V)_{\phi^*(x)} \arrow[r, "(\phi^*(\alpha))_*"'] & X(V)_{\phi^*(y)} \end{tikzcd}$$

commute for each $\alpha : x \to y$ of $\mathrm{Mor}(G)$ and all $\phi : V \to U$ of $\mathscr{C}$.

From this alternative point of view, it's easy to see that a G-functor X defines a natural simplicial (pre)sheaf homomorphism

$$p : \underrightarrow{\mathrm{holim}}_G\, X \to BG.$$

One makes the construction sectionwise.

Remark 9.12. The homotopy colimit construction for G-functors is a direct generalization of the Borel construction for sheaves Y with actions by sheaves of groups H: the simplicial sheaf $EH \times_H Y$ is the homotopy colimit $\underrightarrow{\mathrm{holim}}_H Y$.

Say that a G-functor X is a *G-torsor* if the canonical map

$$\underrightarrow{\mathrm{holim}}_G\, X \to *$$

is a local weak equivalence.

A *morphism* $f : X \to Y$ of G-torsors is a natural transformation of G-functors, namely a sheaf morphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{Ob}(G) & \end{tikzcd}$$

fibred over $\mathrm{Ob}(G)$ which respects the multiplication maps. Write $G-\mathbf{tors}$ for the category of G-torsors and the natural transformations between them.

The diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"'] & \underrightarrow{\mathrm{holim}}_G X \arrow[d, "p"] \\ \mathrm{Ob}(G) \arrow[r] & BG \end{tikzcd}$$

is homotopy cartesian in each section by Quillen's Theorem B [24, IV.5.2] (more specifically, Lemma 5.7), since G is a (pre)sheaf of groupoids, and is therefore homotopy cartesian in simplicial sheaves. It follows that a morphism $f : X \to Y$ of G-torsors specializes to a weak equivalence $X \to Y$ of constant simplicial sheaves, which is therefore an isomorphism.

We therefore have the following generalization of Lemma 9.4:

Lemma 9.13. *Suppose that G is a sheaf of groupoids. The the category $G-$* **tors** *of G-torsors is a groupoid.*

Every G-torsor X has an associated cocycle

$$* \xleftarrow{\simeq} \underrightarrow{\mathrm{holim}}_G X \xrightarrow{p} BG,$$

called the *canonical cocycle*, and this association defines a functor

$$\phi : G-\mathbf{tors} \to h(*, BG)$$

taking values in the simplicial sheaf cocycle category.

Now suppose given a cocycle

$$* \xleftarrow{\simeq} Y \xrightarrow{g} BG$$

in simplicial sheaves and form the pullback diagrams

$$\begin{tikzcd} \mathrm{pb}(Y)(U)_x \arrow[r] \arrow[d] & Y(U) \arrow[d, "g"] \\ B(G(U)/x) \arrow[r] & BG(U) \end{tikzcd}$$

of simplicial sets for each $x \in \mathrm{Ob}(G(U))$ and $U \in \mathscr{C}$. Set

$$\mathrm{pb}(Y)(U) = \bigsqcup_{x \in \mathrm{Ob}(G(U))} \mathrm{pb}(Y)(U)_x.$$

Then the resulting simplicial presheaf map $\mathrm{pb}(Y) \to \mathrm{Ob}(G)$ defines a simplicial presheaf valued functor on G. There is a sectionwise weak equivalence

$$\underrightarrow{\mathrm{holim}}_G \ \mathrm{pb}(Y) \to Y \simeq *$$

by Lemma 9.9, and the diagram

$$\begin{tikzcd} \mathrm{pb}(Y) \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}_G \ \mathrm{pb}(Y) \arrow[d] \\ \mathrm{Ob}(G) \arrow[r] & BG \end{tikzcd}$$

is sectionwise homotopy cartesian. It follows that the natural transformation

$$\mathrm{pb}(Y) \to \tilde{\pi}_0(Y)$$

of simplicial presheaf-valued functors on G is a local weak equivalence. In summary, we have local weak equivalences

$$\underrightarrow{\mathrm{holim}}_G \ \tilde{\pi}_0 \, \mathrm{pb}(Y) \simeq \underrightarrow{\mathrm{holim}}_G \ \mathrm{pb}(Y) \simeq Y \simeq *,$$

so that the sheaf-valued functor $\tilde{\pi}_0 \, \mathrm{pb}(Y)$ on G is a G-torsor. These constructions are functorial on $h(*,BG)$ and there is a functor

$$\psi : h(*,BG) \to G - \mathbf{tors}.$$

Theorem 9.14. *The functors ϕ and ψ induce a homotopy equivalence*

$$B(G - \mathbf{tors}) \simeq Bh(*,BG).$$

Corollary 9.15. *The functors ϕ and ψ induce a bijection*

$$\pi_0(G - \mathbf{tors}) \cong [*,BG].$$

There are multiple possible proofs of Corollary 9.15 (see also [42]), but it is convenient here to use a trick for diagrams of simplicial sets which are indexed by groupoids.

Suppose that Γ is a small groupoid, and let $s\mathbf{Set}^{\Gamma}$ be the category of Γ-diagrams in simplicial sets. Let $s\mathbf{Set}/B\Gamma$ be the category of simplicial set morphisms $Y \to B\Gamma$. The homotopy colimit defines a functor

$$\underrightarrow{\mathrm{holim}}_{\Gamma} : s\mathbf{Set}^{\Gamma} \to s\mathbf{Set}/B\Gamma.$$

This functor sends a diagram $X : \Gamma \to s\mathbf{Set}$ to the canonical map $\underrightarrow{\mathrm{holim}}_{\Gamma} X \to B\Gamma$. On the other hand, given a simplicial set map $Y \to B\Gamma$, the collection of pullback diagrams

$$\begin{tikzcd} \mathrm{pb}(Y)_x \arrow[r] \arrow[d] & Y \arrow[d] \\ B(\Gamma/x) \arrow[r] & B\Gamma \end{tikzcd}$$

defines an Γ-diagram $\mathrm{pb}(Y) : \Gamma \to s\mathbf{Set}$ which is functorial in $Y \to B\Gamma$.

Lemma 9.16. *Suppose that Γ is a groupoid. Then the functors*

$$\mathrm{pb} : s\mathbf{Set}/B\Gamma \leftrightarrows s\mathbf{Set}^{\Gamma} : \underrightarrow{\mathrm{holim}}_{\Gamma}$$

form an adjoint pair: pb *is left adjoint to* $\underrightarrow{\mathrm{holim}}_{\Gamma}$.

Proof. Suppose that X is a Γ-diagram and that $p : Y \to B\Gamma$ is a simplicial set over $B\Gamma$. Suppose given a natural transformation

$$f : \mathrm{pb}(Y)_n \to X_n.$$

and let x be an object of Γ. Then an element of $(\mathrm{pb}(Y)_x)_n$ can be identified with a pair

$$(x, a_0 \to \cdots \to a_n \xrightarrow{\alpha} x)$$

where the string of arrows is in Γ and $p(x)$ is the string $a_0 \to \dots a_n$. Then f is uniquely determined by the images of the elements

$$f(x, a_0 \to \cdots \to a_n \xrightarrow{1} a_n)$$

in $X_n(a_n)$. Since Γ is a groupoid, an element $y \in X(a_n)$ uniquely determines an element

$$(y_0, a_0) \to (y_1, a_1) \to \dots (y_n, a_n)$$

with $y_n = y$. It follows that there is a natural bijection

$$\hom_{\Gamma}(\mathrm{pb}(Y)_n, X_n) \cong \hom_{B\Gamma_n}(Y_n, (\underrightarrow{\mathrm{holim}}_{\Gamma} X)_n).$$

Extend simplicially to get the adjunction isomorphism

$$\hom_{\Gamma}(\mathrm{pb}(Y), X) \cong \hom_{B\Gamma}(Y, \underrightarrow{\mathrm{holim}}_{\Gamma} X).$$

Proof (Proof of Theorem 9.14). It follows from Lemma 9.16 that the functor ψ is left adjoint to the functor ϕ.

Suppose that H is a groupoid and that $x \in \mathrm{Ob}(H)$. The groupoid H/x has a terminal object and hence determines a cocycle

$$* \xleftarrow{\simeq} B(H/x) \to BH.$$

If $a \in \mathrm{Ob}(H)$ then in the pullback diagram

$$\begin{tikzcd}
\mathrm{pb}(B(H/x))(a) \arrow[r] \arrow[d] & B(H/x) \arrow[d] \\
B(H/a) \arrow[r] & BH
\end{tikzcd}$$

the object $\mathrm{pb}(B(H/x))(a)$ is the nerve of a groupoid whose objects are the diagrams

$$a \xleftarrow{\alpha} b \xrightarrow{\beta} x$$

in H, and whose morphisms are the diagrams

$$\begin{tikzcd}
& b \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd] & \\
a & & x \\
& b' \arrow[ul, "\alpha'"] \arrow[ur, "\beta'"'] &
\end{tikzcd}$$

In the presence of such a picture, $\beta \cdot \alpha^{-1} = \beta' \cdot (\alpha')^{-1}$. There are uniquely determined diagrams

$$\begin{tikzcd}
& b \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd] & \\
a & & x \\
& a \arrow[ul, "1"] \arrow[ur, "\beta\cdot\alpha^{-1}"'] &
\end{tikzcd}$$

for each object $a \xleftarrow{\alpha} b \xrightarrow{\beta} x$. It follows that there is a natural bijection

$$\pi_0\, \mathrm{pb}(B(H/x)(a) \cong \hom_H(a,x)$$

and that

$$\mathrm{pb}(B(H/x))(a) \to \pi_0\, \mathrm{pb}(B(H/x))(a)$$

is a natural weak equivalence.

It also follows that there are weak equivalences

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}_{a\in H}\, \mathrm{pb}(B(H/x))(a) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & B(H/x) \simeq * \\
\underrightarrow{\mathrm{holim}}_{a\in H} \hom_H(a,x) &
\end{tikzcd}$$

so that the functor $a \mapsto \hom_H(a,x)$ defines an H-torsor. Here, the function

$$\beta_* : \hom_H(a,x) \to \hom_H(b,x)$$

induced by $\beta : a \to b$ is precomposition with β^{-1}.

To put it a different way, each $x \in H$ determines a H-torsor $a \mapsto \hom_H(a,x)$, which we'll call $\hom_H(\ ,x)$ and there is a functor

$$H \to H - \mathbf{tors}$$

which is defined by $x \mapsto \hom_H(\ ,x)$.

Observe that the maps $\hom_H(\ ,x) \to Y$ classify elements of $Y(x)$ for all functors $Y : H \to \mathbf{Set}$.

In general, every global section x of a sheaf of groupoids G determines a G-torsor $\hom_G(\ ,x)$ which is constructed sectionwise according to the recipe above. In particular, this is the torsor associated by the pullback construction to the cocycle

$$* \xleftarrow{\simeq} B(G/x) \to BG.$$

The torsors $\hom_G(\ ,x)$ are the *trivial torsors* for the sheaf of groupoids G. There is a functor

$$j : \Gamma_* G \to G - \mathbf{tors}$$

which is defined by $j(x) = \hom_G(\ ,x)$.

Observe that torsor (iso)morphisms

$$\begin{tikzcd} \hom_G(\ ,x) \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & \mathrm{Ob}(G) & \end{tikzcd}$$

or *trivializations* are in bijective correspondence with global sections of X which map to $x \in \mathrm{Ob}(G)$ under the structure map $X \to \mathrm{Ob}(G)$.

Remark 9.17. We have not discussed the size of the objects involved in Theorem 9.14. The statement makes no sense unless the cocycle and torsor categories are small in some sense. However, G-torsors are locally isomorphic to some $\hom_G(\ ,x)$ since sections of $\mathrm{Ob}(G)$ lift locally to X, so that there is some cardinal α which bounds all G-torsors (and is also an upper bound on $|\mathrm{Mor}(\mathscr{C})|$), and the category $G - \mathbf{tors}$ can be taken to be small. On the other hand, the canonical cocycle functor

$$\phi : G - \mathbf{tors} \to h(*, BG)$$

takes values in the full subcategory $h(*, BG)_\gamma$ of cocycles

$$* \xleftarrow{\simeq} Y \to BG$$

of cocycles which are bounded by γ in the sense that $|Y| < \gamma$ provided that $\alpha \leq \gamma$. Thus the correct statement of Theorem 9.14 is to assert that there is an infinite cardinal α so that there are homotopy equivalences

$$B(G - \mathbf{tors}) \simeq Bh(*, BG)_\gamma.$$

for all cardinals $\gamma \geq \alpha$. In particular, the map

$$Bh(*,BG)_\alpha \to Bh(*,BG)_\gamma$$

is a weak equivalence for all $\gamma \geq \alpha$ for some infinite cardinal α.

See Proposition 5.36 and Lemma 5.39.

These constructions restrict well. If $\phi : V \to U$ is a morphism of the underlying site $\mathscr{C}$ then composition with ϕ defines a functor

$$\phi_* : \mathscr{C}/V \to \mathscr{C}/U,$$

and composition with ϕ_* determines a restriction functor

$$\phi^* : \mathbf{Pre}(\mathscr{C}/U) \to \mathbf{Pre}(\mathscr{C}/V)$$

which takes $F|_U$ to $F|_V$ for any presheaf F on $\mathscr{C}$. All restriction functors take sheaves to sheaves and are exact. Thus, ϕ^* takes a $G|_U$-torsor to a $G|_V$ torsor. In particular,

$$\phi^* \hom_{G|_U}(\ ,x) = \hom_{G|_V}(\ ,x_V)$$

for all $x \in G(U)$. The functor ϕ^* also preserves cocycles.

It follows that there is a presheaf of groupoids $G - \mathbf{Tors}$ on the site $\mathscr{C}$ with

$$G - \mathbf{Tors}(U) = G|_U - \mathbf{tors}$$

and a presheaf of categories $\mathbb{H}(*,BG)$ with

$$\mathbb{H}(*,BG)(U) = h(*,BG|_U).$$

and there are functors

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[dr] & G - \mathbf{Tors} \arrow[d, "\phi"] \\ & \mathbb{H}(*,BG) \end{tikzcd}$$

where ϕ induces a sectionwise weak equivalence

$$B(G - \mathbf{Tors}) \xrightarrow{\simeq} B\mathbb{H}(*.BG)$$

by Theorem 9.14, and the displayed map is defined by sending an object $x \in G(U)$ to the cocycle $B(G|_U/x) \to BG|_U$.

The images $\hom(\ ,x)$ of the functor $j : G \to G - \mathbf{Tors}$ are the *trivial torsors*, and maps (isomorphisms) $\hom(\ ,x) \to X$ of G-torsors are global sections of X. Every G-torsor X has sections along some cover, since $\underrightarrow{\mathrm{holim}}_G X \to *$ is a local weak equivalence, so every G-torsor is locally trivial.

9.3 Stacks and homotopy theory

Write $\mathbf{Pre}(\mathbf{Gpd}(\mathscr{C}))$ for the category of presheaves of groupoids on a small site $\mathscr{C}$.

Say that a morphism $f : G \to H$ of presheaves of groupoids is a *local weak equivalence* (respectively *injective fibration*) if and only if the induced map $f_* : BG \to BH$ is a local weak equivalence (respectively injective fibration). A morphism $i : A \to B$ of presheaves of groupoids is a *cofibration* if it has the left lifting property with respect to all trivial fibrations.

The fundamental groupoid functor $X \mapsto \pi(X)$ is left adjoint to the nerve functor. It follows that every cofibration $A \to B$ of simplicial presheaves induces a cofibration $\pi(A) \to \pi(B)$ of presheaves of groupoids. The class of cofibrations $A \to B$ is closed under pushout along arbitrary morphisms $A \to G$, because cofibrations are defined by a left lifting property.

There is a function complex construction for presheaves of groupoids: the simplicial set $\mathbf{hom}(G,H)$ has for n-simplices all morphisms

$$\phi : G \times \pi(\Delta^n) \to H.$$

There is a natural isomorphism

$$\mathrm{hom}(G,H) \cong \mathrm{hom}(BG,BH),$$

which sends the simplex ϕ to the composite

$$BG \times \Delta^n \xrightarrow{1\times\eta} BG \times B\pi(\Delta^n) \cong B(G \times \pi(\Delta^n)) \xrightarrow{\phi_*} BH.$$

The following result appears in [28]:

Proposition 9.18. *With these definitions, the category* $\mathbf{Pre}(\mathbf{Gpd}(\mathscr{C}))$ *satisfies the axioms for a right proper closed simplicial model category.*

Proof. The injective model structure for the category $s\mathbf{Pre}(\mathscr{C})$ is cofibrantly generated. It follows easily that every morphism $f : G \to H$ has a factorization

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & H \end{tikzcd}$$

such that j is a cofibration and p is a trivial fibration.

The other factorization axiom can be proved the same way, provided one knows that if $i : A \to B$ is a trivial cofibration of simplicial presheaves and the diagram

$$\begin{tikzcd} \pi(A) \arrow[r] \arrow[d, "i_*"'] & G \arrow[d, "i'"] \\ \pi(B) \arrow[r] & H \end{tikzcd}$$

is a pushout, then the map i' is a local weak equivalence. But one can prove the corresponding statement for ordinary groupoids, and the general case follows by a Boolean localization argument (exercise).

The claim is proved for ordinary groupoids by observing that in all pushout diagrams

$$\begin{tikzcd} \pi(\Lambda^n_k) \arrow[r] \arrow[d, "i_*"'] & G \arrow[d, "i'"] \\ \pi(\Delta^n) \arrow[r] & H \end{tikzcd}$$

the map i_* is an isomorphism for $n \geq 2$ and is the inclusion of a strong deformation retraction if $n = 1$. The classes of isomorphisms and strong deformation retractions are both closed under pushout in the category of groupoids.

All other closed model axioms are trivial to verify, as is right properness. The simplicial model axiom **SM7** has an elementary argument, which ultimately follows from the fact that the fundamental groupoid functor preserves products.

One can make the same definitions for sheaves of groupoids: say that a map $f : G \to H$ of sheaves of groupoids is a *local weak equivalence* (respectively *injective fibration*) if the associated simplicial sheaf map $f_* : BG \to BH$ is a local weak equivalence (respectively injective fibration). *Cofibrations* are defined by a left lifting property, as before.

Write $\mathbf{Shv}(\mathbf{Gpd}(\mathscr{C}))$ and observe that the forgetful functor i and associated sheaf functor L^2 induce an adjoint pair

$$L^2 : \mathbf{Pre}(\mathbf{Gpd}(\mathscr{C})) \leftrightarrows \mathbf{Shv}(\mathbf{Gpd}(\mathscr{C})) : i$$

According to the definitions, the forgetful functor i preserves fibrations and trivial fibrations. Moreover, the canonical map $\eta : BG \to iL^2BG$ is always a local weak equivalence. The method of proof of Proposition 9.18 and formal nonsense now combine to prove the following

Proposition 9.19. *1) With these definitions, the category* $\mathbf{Shv}(\mathbf{Gpd}(\mathscr{C}))$ *of sheaves of groupoids satisfies the axioms for a right proper closed simplicial model category.*

2) The adjoint pair

$$L^2 : \mathbf{Pre}(\mathbf{Gpd}(\mathscr{C})) \leftrightarrows \mathbf{Shv}(\mathbf{Gpd}(\mathscr{C})) : i$$

forms a Quillen equivalence.

The model structures of Proposition 9.18 and 9.19 are the *injective model structures* for presheaves and sheaves of groupoids on a site $\mathscr{C}$, respectively, which model structures are Quillen equivalent.

Part 1) of Proposition 9.19 was first proved in [47]. This was a breakthrough result, in that it enabled the following definition:

Definition 9.20. A sheaf of groupoids H is said to be a *stack* if it satisfies descent for the injective model structure on $\mathbf{Shv}(\mathbf{Gpd}(\mathscr{C}))$.

In other words, H is a stack if and only if every injective fibrant model $j : H \to H'$ is a sectionwise weak equivalence.

Remark 9.21. Classically, stacks are defined to be sheaves of groupoids which satisfy the effective descent condition. The effective descent condition, which is described below, is equivalent to the homotopy theoretic descent condition of Definition 9.20 — this is proved in Proposition 9.26.

Observe that if $j : H \to H'$ is an injective fibrant model in sheaves (or presheaves) of groupoids, then the induced map $j_* : BH \to BH'$ is an injective fibrant model in simplicial presheaves. Thus, H is a stack if and only if the simplicial presheaf BH satisfies descent.

Every injective fibrant object is a stack, because injective fibrant objects satisfy descent. This means that every injective fibrant model $j : G \to H$ of a sheaf of groupoids G is a *stack completion.* . This model j can be constructed functorially, since the injective model structure on $\mathbf{Shv}(\mathbf{Gpd}(\mathscr{C}))$ is cofibrantly generated. We can therefore speak unambiguously about "the" stack completion of a sheaf of groupoids G — the stack completion is also called the *associated stack*.

Similar definitions can also be made for presheaves of groupoids. This means, effectively, that stacks are identified with homotopy types of presheaves or sheaves of groupoids, within the respective injective model structures.

Some of the most common examples of stacks come from group actions. Suppose that $G \times X \to X$ is an action of a sheaf of groups G on a sheaf X. Then the Borel construction $EG \times_G X$ is the nerve of a sheaf of groupoids $E_G X$. The stack completion

$$j : E_G X \to [X/G]$$

is called the *quotient stack*.

A *G-torsor over X* is a G-equivariant map $P \to X$ where P is a G-torsor. A morphism of G-torsors over X is a commutative diagram

$$\begin{tikzcd} P \arrow[rr, "\theta"] \arrow[dr] & & P' \arrow[dl] \\ & X & \end{tikzcd}$$

of G-equivariant morphisms, where P and P' are G-torsors. Write $G - \mathbf{tors}/X$ for the corresponding groupoid.

If $P \to X$ is a G-torsor over X, then the induced map of Borel constructions

$$* \xleftarrow{\simeq} EG \times_G P \to EG \times_G X$$

is an object of the cocycle category

$$h(*, EG \times_G X),$$

and the assignment is functorial. Conversely, if the diagram

$$* \xleftarrow{\simeq} U \to EG \times_G X$$

is a cocycle, then the induced map

$$\tilde{\pi}_0 \operatorname{pb}(U) \to \tilde{\pi}_0 \operatorname{pb}(EG \times_G X) \xrightarrow[\cong]{\varepsilon} X$$

is a G-torsor over X. Here, as above, the pullback functor pb is defined over the canonical map $EG \to BG$.

The functors

$$\tilde{\pi}_0 \operatorname{pb} : h(*, EG \times_G X) \leftrightarrows G - \mathbf{tors}/X : EG \times_G ?$$

are adjoint, and we have proved

Lemma 9.22. *There is a weak equivalence*

$$B(G - \mathbf{tors}/X) \simeq Bh(*, EG \times_G X).$$

In particular, there is an induced bijection

$$\pi_0(G - \mathbf{tors}/X) \cong [*, EG \times_G X].$$

Lemma 9.22 was proved by a different method in [39]. There is a generalization of this result, having essentially the same proof, for the homotopy colimit $\underrightarrow{\operatorname{holim}}_G X$ of a diagram X on a sheaf of groupoids G. See [44].

The sheaves of groupoids G and H are said to be *Morita equivalent* if there is a diagram

$$G \xleftarrow{p} K \xrightarrow{q} H$$

of morphisms such that the induced maps p_* and q_* in the diagram

$$BG \xleftarrow{p_*} BK \xrightarrow{q_*} BH$$

are local trivial fibrations of simplicial sheaves.

Clearly, if G and H are Morita equivalent then they are weakly equivalent for the injective model structure.

Conversely, if $f : G \to H$ is a local weak equivalence, take the cocycle

$$G \xrightarrow{(1,f)} G \times H$$

and find a factorization

$$\begin{tikzcd} G \arrow[r, "j"] \arrow[dr, "(1,f)"'] & K \arrow[d, "(p_1,p_2)"] \\ & G \times H \end{tikzcd}$$

such that j is a local weak equivalence and (p_1,p_2) is an injective fibration. Then the induced map

$$BK \xrightarrow{(p_{1*},p_{2*})} BG \times BH$$

is an injective hence local fibration, and the projection maps $BG \times BH \to BG$ and $BG \times BH \to BH$ are local fibrations since BG and BH are locally fibrant. It follows that the maps

$$G \xleftarrow{p_1} K \xrightarrow{p_2} H$$

define a Morita equivalence.

We have shown the following:

Lemma 9.23. *Suppose that G and H are sheaves of groupoids. Then G and H are locally weakly equivalent if and only if they are Morita equivalent.*

Categories of cocycles and torsors can both be used to construct models for the associated stack. The precise statement appears in Corollary 9.25, which is a corollary of the proof of the following:

Proposition 9.24. *Suppose that G is a sheaf of groupoids on a small site $\mathscr{C}$. Then the induced maps*

$$\begin{tikzcd} BG \arrow[r, "j_*"] \arrow[dr] & B(G-\mathbf{Tors}) \arrow[d, "\phi_*"] \\ & B\mathbb{H}(*,BG) \end{tikzcd}$$

are local weak equivalences of simplicial presheaves.

Proof. Suppose, first of all, that H is an injective fibrant sheaf of groupoids. We show that the morphisms

$$\begin{tikzcd} BH \arrow[r, "j_*"] \arrow[dr] & B(H-\mathbf{Tors}) \arrow[d, "\phi_*"] \\ & B\mathbb{H}(*,BH) \end{tikzcd}$$

are sectionwise weak equivalences of simplicial sheaves.

The map ϕ_* is a sectionwise equivalence for all sheaves of groupoids H by Theorem 9.14. It is an exercise to show that the morphism j is fully faithful in all sections, again for all sheaves of groupoids H.

Thus, it suffices to show that all maps

$$j_* : \pi_0 BH(U) \to \pi_0 B(H - \mathbf{Tors})(U)$$

is surjective for all $U \in \mathscr{C}$ under the assumption that H is injective fibrant. For this, we can assume that the site $\mathscr{C}$ has a terminal object t and show that the map

$$\pi_0 BH(t) \to \pi_0 B\mathbb{H}(*, BH)(t) = \pi_0 Bh(*, BH)$$

is surjective.

In every cocycle

$$* \xleftarrow{s} U \xrightarrow{f} BH$$

the map s is a local weak equivalence, so there is a homotopy commutative diagram

$$\begin{tikzcd} U \arrow[d, "s"'] \arrow[dr, "f"] & \\ * \arrow[r, "x"'] & BH \end{tikzcd}$$

since BH is injective fibrant. This means that the cocycles (s, f), (s, xs) and $(1, x)$ are all in the same path component of $h(*, BH)$.

For the general case, suppose that $i : G \to H$ is an injective fibrant model for G. In the diagram

$$\begin{tikzcd} BG \arrow[r, "i_*"] \arrow[d, "j"'] & BH \arrow[d, "\simeq"', "j"] \\ B(G - \mathbf{Tors}) \arrow[r] \arrow[d, "\phi_*"', "\simeq"] & B(H - \mathbf{Tors}) \arrow[d, "\simeq"', "\phi_*"] \\ B\mathbb{H}(*, BG) \arrow[r, "\simeq", "i_*"'] & B\mathbb{H}(*, BH) \end{tikzcd}$$

the indicated maps are sectionwise weak equivalences: use the paragraphs above for the vertical maps on the right, Corollary 5.44 for the bottom i_* (see also Remark 9.17), and Theorem 9.14 for ϕ_*. The map $i_* : BG \to BH$ is a local weak equivalence since the map i is an injective fibrant model. It follows that the map $j_* : BG \to B(G - \mathbf{Tors})$ is a local weak equivalence.

Corollary 9.25. *Suppose that G is a sheaf of groupoids. Then the maps*

$$j : G \to G - \mathbf{Tors}$$

and

$$\phi \cdot j : G \to \mathbb{H}(*, BG)$$

are models for the stack completion of G.

Suppose that $R \subset \hom(\ ,U)$ is a covering sieve, and also write R for the full subcategory on $\mathscr{C}/U$ whose objects are the members $\phi : V \to U$ of the sieve. Following Giraud [22], an *effective descent datum* $x : R \to G$ on the sieve R consists of

1) objects $x_\phi \in G(V)$, one for each object $\phi : V \to U$ of R, and
2) morphisms $x_\phi \xrightarrow{\alpha_*} \alpha^*(x_\psi)$ in $G(V)$, one for each morphism

$$\begin{tikzcd} V \arrow[rr, "\alpha"] \arrow[dr, "\phi"'] & & W \arrow[dl, "\psi"] \\ & U & \end{tikzcd}$$

of R,

such that the diagram

$$\begin{tikzcd} x_\phi \arrow[r, "\alpha_*"] \arrow[d, "(\beta\alpha)_*"'] & \alpha^*(x_\psi) \arrow[d, "\alpha^*(\beta_*)"] \\ (\beta\alpha)^*(x_\zeta) \arrow[r, "="'] & \alpha^*\beta^*(x_\zeta) \end{tikzcd}$$

commutes for each composable pair of morphisms

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[dr, "\phi"'] & W \arrow[r, "\beta"] \arrow[d, "\psi"] & W' \arrow[dl, "\zeta"] \\ & U & \end{tikzcd}$$

in R.

There is a functor $R \to \mathbf{Pre}(\mathscr{C})$ which takes an object $\phi : V \to U$ to the representable functor $\hom(\ ,V)$. The corresponding translation object E_R is a presheaf of categories on $\mathscr{C}$: the presheaf of objects for E_R is the disjoint union

$$\bigsqcup_{\phi : V \to U} \hom(\ ,V)$$

and the presheaf of morphisms for E_R is the disjoint union

$$\bigsqcup_{\begin{tikzcd}[sep=small] V \arrow[rr, "\alpha"] \arrow[dr, "\phi"'] & & W \arrow[dl, "\psi"] \\ & U & \end{tikzcd}} \hom(V).$$

which is indexed over the morphisms of R.

Then an effective descent datum $x : R \to G$ can be identified with a functor $x : E_R \to G$ of presheaves of categories. A *morphism* of effective descent data is a natural transformation of such functors. Write $\hom(E_R, G)$ for the corresponding groupoid of effective descent data on the sieve R.

Any refinement $S \subset R$ of covering sieves induces a restriction functor

$$\hom(E_R, G) \to \hom(E_S, G),$$

and in particular the inclusion $R \subset \hom(\ ,U)$ induces a functor

$$G(U) \to \hom(E_R, G). \tag{9.5}$$

One says that the sheaf of groupoids G satisfies the *effective descent* condition if an only if the map (9.5) is an equivalence of groupoids for all covering sieves $R \subset \hom(\ ,U)$ and all objects U of the site $\mathscr{C}$.

The effective descent condition is the classical criterion for a sheaf of groupoids to be a stack, and we have the following:

Proposition 9.26. *A sheaf of groupoids G is a stack if and only if it satisfies the effective descent condition.*

Proof. Suppose that G is a stack. The effective descent condition is an invariant of sectionwise equivalence of groupoids, so it suffices to assume that G is injective fibrant. The nerve of the groupoid $\hom(E_R, G)$ may be identified up to isomorphism with the function complex $\hom(BE_R, BG)$. There is a canonical local weak equivalence $BE_R \to U$ (see Lemma 9.27 below), and so the induced map

$$BG(U) \to \hom(BE_R, BG)$$

is a weak equivalence of simplicial sets. This means, in particular, that the homomorphism

$$G(U) \to \hom(E_R, G)$$

is an equivalence of groupoids. It follows that G satisfies the effective descent condition.

Suppose, conversely, that the sheaf of groupoids G satisfies the effective descent condition, and let $j : G \to H$ be an injective fibrant model in sheaves of groupoids. We must show that the map j is a sectionwise equivalence of groupoids.

The map j is fully faithful in all sections since it is a local equivalence between sheaves of groupoids. It therefore suffices to show that the induced map $\pi_0 G(U) \to \pi_0 H(U)$ is surjective for each $U \in \mathscr{C}$. In view of the commutativity of the diagram

$$\begin{tikzcd}
\pi_0 G(U) \arrow[r] \arrow[d] & \pi_0 h(\tilde{U}, BG) \arrow[d, "\cong"] \\
\pi_0 H(U) \arrow[r, "\cong"'] & \pi_0 h(\tilde{U}, BH)
\end{tikzcd}$$

it further suffices to show that each canonical function

$$\pi_0 G(U) \to \pi_0 h(\tilde{U}, BG)$$

is surjective. Here, $\tilde{U}$ is the sheaf which is associated to the representable presheaf $\hom(\ ,U)$.

Suppose that

$$\tilde{U} \underset{\simeq}{\xleftarrow{f}} V \xrightarrow{g} BG$$

is a cocycle in simplicial sheaves. There is a diagram

$$\begin{tikzcd} & V \arrow[dl, "f"'] \arrow[d] \arrow[dr, "g"] & \\ \tilde{U} & & BG \\ & \check{C}(V_0/\tilde{U}) \arrow[ul] \arrow[ur, "g_*"'] & \end{tikzcd}$$

where $\check{C}(V_0/\tilde{U})$ is the nerve of the fundamental groupoid sheaf $\tilde{\pi}(V)$ of the simplicial sheaf V. The fundamental groupoid sheaf $\tilde{\pi}(V)$ can be identified up to isomorphism with the Čech groupoid $C(p)$ for the local epimorphism $p : V_0 \to \tilde{U}$ (Example 4.18). In effect, the canonical map $\tilde{\pi}(V) \to C(p)$ is fully faithful and is an isomorphism on objects in all sections.

Let $R \subset \hom(\ ,U)$ be the covering sieve of all maps $\phi : V \to U$ which lift to V_0, and pick a lifting $\sigma_\phi : V \to V_0$ for each such ϕ. The morphisms σ_ϕ define a morphism

$$\sigma : \bigsqcup_{V \xrightarrow{\phi} U \in R} V \to V_0.$$

If $\alpha : \phi \to \psi$ is a morphism of R then there is a diagram

$$\begin{tikzcd} V \arrow[rr, "\alpha"] \arrow[dr, "\alpha_*"] \arrow[dd, "\sigma_\phi"'] & & W \arrow[dd, "\sigma_\psi"] \\ & V_0 \times_U V_0 \arrow[dl] \arrow[dr] & \\ V_0 \arrow[dr] & & V_0 \arrow[dl] \\ & U & \end{tikzcd}$$

The collection of these maps α_* defines a morphism

$$\bigsqcup_{\phi \xrightarrow{\alpha} \psi} V \to V_0 \times_U V_0,$$

and we have defined a functor $\sigma : E_R \to C(p)$. There is a corresponding diagram

$$\begin{tikzcd} & \check{C}(V_0/\tilde{U}) \arrow[dl] \arrow[dr, "g_*"] & \\ \tilde{U} & & BG \\ & BE_R \arrow[ul] \arrow[uu, "\sigma"] \arrow[ur, "g'"'] & \end{tikzcd}$$

Finally, the assumption that G satisfies effective descent means that there is a homotopy commutative diagram

$$\begin{tikzcd} BE_R \arrow[r, "g'"] \arrow[d] & BG \\ \tilde{U} \arrow[ur, dotted] & \end{tikzcd}$$

and it follows that the original cocycle (f,g) is in the path component of a cocycle of the form

$$\tilde{U} \xleftarrow{1} \tilde{U} \to BG.$$

Lemma 9.27. *Suppose that $R \subset \hom(\ ,U)$ is a covering sieve. Then the canonical map*

$$BE_R \to U$$

of simplicial presheaves is a local weak equivalence.

Proof. Suppose that $W \in \mathscr{C}$, and consider the induced map of W-sections

$$\bigsqcup_{\phi_0 \to \dots \to \phi_n} \hom(W, V_0) \to \hom(W, U).$$

The fibre F_ϕ over a fixed morphism $\phi : W \to U$ is the nerve of the category of factorizations

$$\begin{tikzcd} & V \arrow[d, "\psi"] \\ W \arrow[ur] \arrow[r, "\phi"'] & U \end{tikzcd}$$

of ϕ with $\psi \in R$. If $\phi : W \to U$ is a member of R and then this category is non-empty and has an initial object, namely the picture

$$\begin{tikzcd} & W \arrow[d, "\phi"] \\ W \arrow[ur, "1"] \arrow[r, "\phi"'] & U \end{tikzcd}$$

The fibre F_ϕ is empty if ϕ is not a member of R.

In all cases, there is a covering sieve $S \subset \hom(\ ,W)$ such that $\phi \cdot \psi$ is in R for all $\psi \in S$.

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Index